Proof Round 2023-2024

Problem 1. **[5]** Alice and Bob play a game on a $m \times n$ chessboard ($m \geq 1, n \geq 3$). They alternate turns, with Alice going first. On each turn, a player may place either a domino, covering exactly 2 squares on the chessboard, or a 3×1 tromino, covering exactly 3 squares on the chessboard, such that the newly placed piece does not overlap any previously placed pieces. Pieces may be placed either horizontally or vertically. Whoever cannot play a legal move at some point loses. For each (m,n), find who has the winning strategy.

Problem 2. **[10]** Let X be a finite set, and let $\mathscr{P}(X)$ be the *power set* of X; that is, $\mathscr{P}(X)$ is the set of all subsets of X. For instance, if $X = \{a,b,c\}$, then

$$\mathscr{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}.$$

In particular, if $|X|$ is the cardinality of X, then the cardinality of $\mathscr{P}(X)$ is $2^{|X|}$.

A subset $\Sigma \subseteq \mathscr{P}(X)$ is called an *algebra* (on X) if it satisfies the following 3 properties:

(1) (Universal set) $X \in \Sigma$.
(2) (Closed under complement) If $A \in \Sigma$, then $X \setminus A \in \Sigma$.
(3) (Closed under union) If $A_1, A_2 \in \Sigma$, then $A_1 \cup A_2 \in \Sigma$.

For instance, let $X = \{a,b,c\}$. Then, $\mathscr{P}(X)$ itself is an algebra (this is the largest possible algebra on X). As a non-trivial example,

$$\{\emptyset, \{a,b\}, \{c\}, \{a,b,c\}\}$$

is also an algebra on X.

(a) **[1]** Give another two examples of algebras on the set $X = \{a,b,c\}$. (*No proof required*)

(b) **[2]** Let $X = \{1,2,\ldots,k\}$ for some integer $k \geq 3$. Suppose $\Sigma \subseteq \mathscr{P}(X)$ is an algebra such that the sets $\{1,2\}$ and $\{2,3\}$ are in Σ. Show that the one-element sets $\{1\}, \{2\}, \{3\}$ are in Σ.

(c) **[7]** Prove that for any finite set X, the cardinality of any algebra $\Sigma \subseteq \mathscr{P}(X)$ equals 2^n for some positive integer n. *Hint: Consider the following definition. A partition of X (with k parts) is a collection of non-empty subsets $C_1, C_2, \ldots, C_k$ of X such that $\bigcup_{i=1}^{k} C_i = X$, and $C_i \cap C_j = \emptyset$ for every $1 \leq i < j \leq k$. In particular, every element of X lies in exactly one part.*

Problem 3. **[13]** Let $m \geq 1$ be an integer. An *mth root of unity* is a complex number z such that $z^m = 1$. An mth root of unity z is called *primitive* if $z^m = 1$, and $z^d \neq 1$ for every integer $1 \leq d < m$. For instance, $w := \cos(\frac{8\pi}{6}) + i\sin(\frac{8\pi}{6})$ is a 6th root of unity, but not a primitive 6th root of unity. It is, however, a primitive 3rd root of unity.

(a) **[2]** Find, with proof, the product of all primitive mth roots of unity (your answer may depend on m).

(b) **[4]** Prove that the sum of all primitive mth roots of unity is non-zero if and only if m is *squarefree*, i.e. m is not divisible by the square of any prime number.

(c) **[7]** Suppose that m is *squarefree*. Prove that every mth root of unity z can be written as a finite sum or difference of primitive mth roots of unity. In other words, prove that there is some integer $k \geq 1$, primitive mth roots of unity $z_1, z_2, \ldots, z_k$ (not necessarily mutually distinct), and $a_i \in \{-1, +1\}$ for each $1 \leq i \leq k$, such that

$$z = a_1 z_1 + a_2 z_2 + \cdots + a_k z_k.$$

(It turns out that the converse is true as well. That is, if $m \geq 1$ is a not a square-free integer, then there is some mth root of unity z which cannot be written as a finite sum or difference of primitive mth roots of unity. This actually follows from **(b)** and a bit of linear algebra and field theory.)

Problem 4. [7] Professor Tamuz is teaching a class containing some finite number of students (at least 1). Every pair of students in class are either friends or not. Prove that Professor Tamuz may partition the students in the class into some number of groups $N \geq 1$, labelled $1, 2, \ldots, N$, and pick N student representatives, one from each group, such that:

- No pair of students belonging to the same group are friends with each other.
- For any two different groups i and j, the representative of group i is friends with some student belonging to group j.

Problem 5. [10] Say a positive integer x is *power-close* if there exist positive integers $k \geq 4, m \geq 2$ such that there is some $0 \leq i \leq m-1$ with k^m dividing $x - i$. Prove that there are infinitely many positive integers that are not power-close.

Problem 6. [15] Find all positive real numbers r, c with $r > 1$ satisfying the following: there exists a positive integer N such that $\lfloor cr^n \rfloor$ is a perfect cube for all positive integers $n \geq N$.

Problem 7. [20] Let ABC be a scalene triangle with incenter I and circumcircle Γ. Let M be the midpoint of arc $\widehat{BC}$ of Γ not containing A, and let X be a point on $\overline{BC}$ such that $IX = XM$. The incircle of ABC is tangent to $\overline{BC}$ at D, and denote by I' the reflection of I over D. Let T lie on Γ such that $\overline{AT} \perp \overline{TI'}$. Finally, $\overline{TD}$ meets Γ again at P. Prove that the circumcircle of triangle IXP is tangent to $\overline{AI}$.