Individual Round 2023-2024 Solutions

Problem 1. A standard analog clock has three hands (seconds, minute, and hour), and the usual 12 hourly labels 1:00, 2:00, ..., 12:00. Each hand rotates around the clock continuously. Assume that this clock records the time perfectly. At 12:00 PM sharp, the three hands of the clock are all pointing in exactly the same direction, at the 12:00 label. After some exact amount of time passes during the same day, the minute and hour hands are again pointing in the exact same direction, while the seconds hand is pointing in a direction closer to the 4:00 label than any of the other hourly labels. How much time, in minutes, has passed? Round your answer to the nearest integer.

Proposed by Brian Yang

Solution: $\boxed{196}$.

Every minute, the minute (resp. hour) hand of the clock moves 6° (resp. 0.5°), as there are 60 minutes (resp. 12 hours, or 720 minutes) in a full revolution of the minute (resp. hour) hand. Let x be the number of minutes until the first time the minute and hour hands "meet" after 12:00; the minute hand has made at least one full revolution, so we must have $6x = 0.5x + 360$. Thus, $x = \frac{720}{11} = 65\frac{5}{11}$ minutes, i.e. the seconds hand is pointing in a direction $\frac{5}{11}$ of a full revolution clockwise of 12:00. By symmetry, the minute and hour hands must meet in periods of $x = 65\frac{5}{11}$ minutes. In particular, on the nth time the minute and hour hands meet, the seconds hand is pointing in a direction $\frac{5n}{11} \pmod 1$ of a full revolution clockwise of 12:00. Now, $\frac{5n}{11} \pmod 1$ may take the values $\{0, \frac{1}{11}, \dots, \frac{10}{11}\}$, so the seconds hand points closer to the 4:00 label than any of the hour labels precisely when $\frac{5n}{11} \equiv \frac{4}{11} \pmod 1$. The values of n "solving" this congruence are given by $n \equiv 3 \pmod{11}$, and only the smallest positive value of $n = 3$ corresponds to the same day. Therefore, the answer is $65\frac{5}{11} \cdot 3$ minutes, rounding to $\boxed{196}$ minutes.

Problem 2. The dreaded pirate captain Jack D. Luffy and his crew found a buried treasure chest on a deserted island, containing gold and silver coins as loot. Monetarily, a single gold coin is worth $Q > 1$ silver coins (Q need not be an integer), and a single silver coin is worth many times one unit of the national currency. The pirates agree to share the spoils fairly; that is, every pirate receives the same amount of loot in monetary value (but the number of gold and silver coins received by each pirate may vary). Jack receives $\frac{1}{6}$ the total number of gold coins and $\frac{1}{10}$ the total number of silver coins from the chest. After counting up his loot again, Jack observes he has received four times as many silver coins as gold coins. What is the sum of all possible values of Q?

Proposed by Brian Yang

Solution: $\boxed{\dfrac{52}{3}}$.

Let s, g be the total *monetary value* of silver and gold contained in the chest, and n the number of pirates (including Jack). Since each pirate receives the same amount of loot, we have

$$n\left(\frac{s}{10} + \frac{g}{6}\right) = s + g.$$

This equation rearranges to $3s(10-n) = 5g(n-6)$. Thus, $n \in \{7, 8, 9\}$. This leaves three cases.

First, assume $n = 7$. Then, $9s = 5g$. Thus, Jack's spoils consists of $\frac{g}{6}$ value in gold and $\frac{5g}{90}$ value in silver. In particular, the ratio of value in gold to value in silver in Jack's spoils is $3:1$. Since these spoils involve four times as many silver coins as gold coins, we deduce $Q = 3 \cdot 4 = 12$, i.e. a single gold coin is worth 12 times a single silver coin.

For the remaining cases, the computation is similar. Next, assume $n = 8$. Then, $3s = 5g$, and Jack's spoils

consists of $\frac{g}{6}$ value in gold and $\frac{g}{6}$ value in silver. The gold to silver value ratio is $1:1$, so $Q = 1 \cdot 4 = 4$.
Finally, assume $n = 9$. Then, $s = 5g$, and Jack's spoils consists of $\frac{g}{6}$ value in gold and $\frac{g}{2}$ value in silver. The gold to silver value ratio is $1:3$, so $Q = \frac{1}{3} \cdot 4 = \frac{4}{3}$.
Hence, the sum of the possible values of Q is

$$12 + 4 + \frac{4}{3} = \boxed{\frac{52}{3}}.$$

Problem 3. A Caltech prefrosh is participating in rotation. There are 8 houses at Caltech: Avery, Blacker, Dabney, Fleming, Lloyd, Page, Ricketts, and Venerable. The prefrosh visits these houses in some order, each of them exactly once. Throughout rotation, the prefrosh maintains a ranking list of all of the houses that the prefosh has visited. After every visit to a house, the prefrosh updates the ranking list by inserting the most recently visited house to either the top or the bottom of the list, each with probability $\frac{1}{2}$, while keeping the order of all previously visited houses the same. Compute the probability that at the end of rotation, third house the prefrosh visited is not ranked fourth or fifth on their list.

Proposed by Justin Lee

Solution: $\boxed{\frac{17}{32}}$.
Towards complementary counting, we shall compute the probability that the 3rd house visited is ranked 4th. Immediately after the 3rd house is visited, there is a $\frac{1}{2}$ probability it is ranked 1st and a $\frac{1}{2}$ probability it is ranked 3rd (last).
In the first case, the 3rd house is ranked 4th at the end of rotation if and only if the prefrosh inserts exactly 3 of the 5 houses after the 3rd house to the top of the list. This occurs with probability $\frac{1}{2^5} \cdot \binom{5}{3} = \frac{10}{32}$. In the second case, the 3rd house is ranked 4th at the end of rotation if and only if the prefrosh inserts exactly 1 of the 5 houses after the 3rd house to the top of the list. This occurs with probability $\frac{1}{2^5} \cdot \binom{5}{1} = \frac{5}{32}$. All in all, the probability the 3rd house is ranked 4th is $\frac{1}{2} \cdot \frac{10}{32} + \frac{1}{2} \cdot \frac{5}{32} = \frac{15}{64}$. By symmetry, the probability the 3rd house is ranked 5th is also $\frac{15}{64}$. Then, the probability the 3rd house is ranked neither 4th nor 5th is $1 - \frac{15}{64} - \frac{15}{64} = \boxed{\frac{17}{32}}$.

Problem 4. Find the number of positive real numbers x such that

$$\log_3(x) = \frac{x+15}{4} - \left\lfloor \frac{x-1}{4} \right\rfloor \qquad \textbf{and} \qquad \lfloor \log_5(x) \rfloor = 2$$

(for any positive real number y, recall that $\lfloor y \rfloor$ is the greatest integer less than or equal to y).

Proposed by Ritvik Teegavarapu and Brian Yang

Solution: $\boxed{11}$.
Let $\{\ \}$ denote fractional part as usual. The first equation rewrites as

$$\log_3(x) = \frac{x+15}{4} - \left\lfloor \frac{x-1}{4} \right\rfloor = 4 + \frac{x-1}{4} - \left\lfloor \frac{x-1}{4} \right\rfloor = 4 + \left\{ \frac{x-1}{4} \right\}.$$

In particular, $4 \le \log_3(x) < 5$, i.e. $81 = 3^4 \le x < 3^5 \le 243$ (by monotonicity of the logarithm). Next, the second equation $\lfloor \log_5(x) \rfloor = 2$ is equivalent to $2 \le \log_5(x) < 3$, i.e. $25 \le x < 125$. Thus, the task is to find all $81 \le x < 125$ in which $\log_3(x) = 4 + \left\{ \frac{x-1}{4} \right\}$.

Observe that the function $\left\{\frac{x-1}{4}\right\}$ is piecewise linear of slope $\frac{1}{4}$ on intervals of the form $[4k+1, 4k+5)$ for integers k. In particular, $4+\left\{\frac{x-1}{4}\right\}=4$ at $x=4k+1$, while the left-sided limit $\lim_{x\to(4k+5)^-}(4+\left\{\frac{x-1}{4}\right\})=5$. Now, note there is some $\delta>0$ in which $\log_3(x)\in[4,5-\delta)$ for all $81\le x<125$. By continuity and the intermediate value theorem, the functions $4+\left\{\frac{x-1}{4}\right\}$ and $\log_3(x)$ take the same value at least once on each of the $\boxed{11}$ intervals $[4k+1,4k+5)=[81,85),[85,89),\dots[121,125)$. In a small open neighborhood of each such interval $[4k+1,4k+5)$, the derivative of $4+\left\{\frac{x-1}{4}\right\}-\log_3(x)$ in x (which is generally $\frac{1}{4}-\frac{1}{x\ln 3}$) is clearly positive, so the functions $4+\left\{\frac{x-1}{4}\right\}$ and $\log_3(x)$ take the same value *at most* once on each interval.

Problem 5. Call a natural number $n>1$ *flavorful* if, for every prime divisor p of n, p^2 is also a divisor of n. Find the largest positive integer that cannot be expressed as the sum of one or more distinct flavorful numbers.

Proposed by Justin Lee

Solution: $\boxed{30}$.

First notice that all multiples of 4 can be expressed as the sum of distinct powers of 2 that are greater than or equal to 4. Now, the three smallest flavorful numbers that are not multiples of 4 are 9, 25, and 27. Any $1 \pmod 4$ number greater than or equal to 9 can be written as 9 plus distinct even flavorful numbers. Moreover, any $3 \pmod 4$ number greater than or equal to 27 can be written as 27 plus distinct even flavorful numbers. Lastly, any $2 \pmod 4$ number greater than or equal to 34 can be written as $9+25$ plus distinct even flavorful numbers. It is clear that 30 cannot be written as the sum of distinct flavorful numbers (for we would need at least two odd numbers in the sum), so the answer is $\boxed{30}$.

Problem 6. Let $\mathscr{R}$ be a right rectangular prism with vertices $A_1,A_2,A_3,A_4,B_1,B_2,B_3,B_4$, where $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two parallel rectangular faces, with $A_1A_2=B_1B_2=3, A_2A_3=B_2B_3=7$, and $\overline{A_1B_1},\overline{A_2B_2},\overline{A_3B_3}$, and $\overline{A_4B_4}$ are mutually parallel edges of $\mathscr{R}$. Suppose that a plane intersects segments $\overline{A_1B_1},\overline{A_2B_2},\overline{A_3B_3}$, and $\overline{A_4B_4}$ at P_1,P_2,P_3, and P_4, respectively, dividing $\mathscr{R}$ into two solids, each with volume exactly $\frac{1}{2}$ that of $\mathscr{R}$. If three of the lengths A_1P_1,A_2P_2,A_3P_3, and A_4P_4 are $3,4$, and 6 in some order, then find the sum of all possible values of the volume of $\mathscr{R}$.

Proposed by Brian Yang

Solution: $\boxed{546}$.

Suppose P_1,P_2,P_3, and P_4 are points on the lines $\overline{A_1B_1},\overline{A_2B_2},\overline{A_3B_3}$, and $\overline{A_4B_4}$, respectively. For the four points P_1,P_2,P_3, and P_4 to be coplanar, we claim it is necessary and sufficient that they form a parallelogram $P_1P_2P_3P_4$. Indeed, it is clear that if they form a parallelogram, then they are coplanar. Conversely, if they are coplanar, then let $P_3'\in\mathbb{R}^3$ be such that $P_1P_2P_3'P_4$ is a parallelogram. Since the orthogonal projections of P_1,P_2, and P_4 onto the plane of $A_1A_2A_3A_4$ are A_1,A_2, and A_4, respectively, it follows P_3' is such that its orthogonal projection onto the plane of $A_1A_2A_3A_4$ is A_3 (since $A_1A_2A_3A_4$ is a rectangle). Thus, P_3' lies on $\overline{A_3B_3}$. Since P_3 also lies in this line, we conclude $P_3=P_3'$, verifying the claim.

Now, assume P_1,P_2,P_3, and P_4 lie on the segments $\overline{A_1B_1},\overline{A_2B_2},\overline{A_3B_3}$, and $\overline{A_4B_4}$, respectively. Put $x_i=P_iA_i$ for $1\le i\le 4$. By a straightforward length-chasing argument (i.e., Cartesian coordinate system), notice $P_1P_2P_3P_4$ is a parallelogram, i.e. the four points are coplanar, if and only if $x_1+x_3=x_2+x_4$. Thus, we must consider the set of all non-negative integers x such that $3,4,6,x$ is a solution to this equation. Out of the four lengths x_1,x_2,x_3,x_4, either the lengths x_1,x_3, or the lengths x_2,x_4, are the smallest and largest, in some order. Hence, if $x<3$, then we must have $x+6=3+4$, yielding $x=1$. If $3\le x\le 6$, then we must have $x+4=3+6$, yielding $x=5$. Finally, if $x>6$, then we must have $x+3=4+6$, yielding $x=7$. This results in 3 4-tuples, which are $(1,3,4,6),(3,4,5,6),(3,4,6,7)$, each representing the lengths x_1,x_2,x_3,x_4 in some order.

For any such 4-tuple, let m,M be the minimum and maximum lengths. In lieu of the volume bisection condition, we claim that $m+M$ is necessarily the length of the lateral edge of $\mathscr{R}$ (i.e., the length of A_1B_1). Indeed, suppose K is the intersection of the diagonals of parallelogram $P_1P_2P_3P_4$. Since the orthogonal projection of K onto the plane of $A_1A_2A_3A_4$ is the center of that rectangle, the length of this orthogonal projection is just the average of x_1,x_2,x_3,x_4, which is exactly $\frac{m+M}{2}$. Then, if K coincides with the center of $\mathscr{R}$, i.e., if $m+M$ is the length of the lateral edge of $\mathscr{R}$, the map $r:\mathscr{R}\to\mathscr{R}$ defined by reflection through K is a bijective isometry, hence is volume preserving. In particular, let $\mathscr{S}_A,\mathscr{S}_B$ be the two solids defined by $\mathscr{R}$ and the plane of $P_1P_2P_3P_4$; then, r maps one of the two solids defined by $\mathscr{R}$ and the plane of $P_1P_2P_3P_4$ onto the second one, so that they have the same volume. Otherwise, K is not the center of $\mathscr{R}$, in which case the image $\mathscr{S}_A$ by r is either properly contained in, or properly contains $\mathscr{S}_B$, so that the two solids have different volumes. Having proven the desired claim, we conclude that the answer is $3\cdot 7\cdot((1+6)+(3+6)+(3+7)) = \boxed{546}$.

Problem 7. League of Legends is a two-team video game, one team playing on the *blue side* and the other playing on the *red side*, where every game results in a win for one team and a loss for the other. The League of Legends teams T1 and JDG play a best-of-five series of games: that is, the two teams play games until one of them has won three games. In the first game, T1 plays on the blue side. In every subsequent game, the team that lost the previous game plays on the blue side. The two teams are equally matched, but "side selection" matters: the probability that the team on the blue side wins any particular game is $\frac{2}{3}$. After the best-of-five series, what is the expected number of games won by the team playing on the blue side?

Proposed by Brian Yang

Solution: $\boxed{\dfrac{238}{81}}$.

We label the two teams A and B, where A plays on the blue side in the first game of the best-of-five. We write outcomes to the best-of-five series as strings of length $3,4$, or 5. The best-of-five series ends in 3 games if and only if one team wins 3 games in a row, i.e. either AAA or BBB occurs. In the former (resp. latter) case, 1 game (resp. 0 games) is won on blue side. Thus, the probability the best-of-five ends in 3 games is $(\frac{2}{3})(\frac{1}{3})^2+(\frac{1}{3})^3=\frac{1}{9}$.

Now consider the case where the best-of-five series ends in 4 games. If A wins the first game, then either ABAA, AABA, or ABBB occurs. These cases correspond to 3, 3, or 2 blue side wins, respectively, yielding a probability of $2\cdot(\frac{2}{3})^3(\frac{1}{3})+(\frac{2}{3})^2(\frac{1}{3})^2=\frac{20}{81}$. Likewise, if B wins the first game, then either BABB, BBAB, or BAAA occurs. These cases correspond to 2, 2, or 1 blue side wins, respectively, yielding a probability of $2\cdot(\frac{2}{3})^2(\frac{1}{3})^2+(\frac{2}{3})(\frac{1}{3})^3=\frac{10}{81}$. All in all, the probability the best-of-five ends in 4 games is $\frac{20}{81}+\frac{10}{81}=\frac{10}{27}$.

In particular, the probability that any one of the first three games is played is 1, the probability that the fourth game is played is $1-\frac{1}{9}=\frac{8}{9}$, and the probability that a fifth game is played is $1-\frac{1}{9}-\frac{10}{27}=\frac{14}{27}$. The expected number of blue side wins contributed by any one of the 5 games, conditioning on the event *the game is played*, is, by definition, $\frac{2}{3}$. On the other hand, the expected number of blue side wins contributed by any one of the 5 games, conditioning on the event the game is not played, is trivially 0. We then conclude by linearity of expectation that the desired expectation value is

$$\frac{2}{3}\left(1+1+1+\frac{8}{9}+\frac{14}{27}\right)=\boxed{\frac{238}{81}}.$$

Problem 8. For a positive integer $k\geq 2$, let $\alpha_k,\ \beta_k,\gamma_k$ be the complex roots (with multiplicity) of the cubic equation $\left(x-\frac{1}{k-1}\right)\left(x-\frac{1}{k}\right)\left(x-\frac{1}{k+1}\right)=\frac{1}{k}$. Determine the value of

$$\sum_{k=2}^{\infty}\frac{\alpha_k\beta_k\gamma_k\cdot(1+\alpha_k)\cdot(1+\beta_k)\cdot(1+\gamma_k)}{k+1}.$$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{27}{16}}$.

Define the polynomial

$$P(x) = \left(x - \frac{1}{k-1}\right) \cdot \left(x - \frac{1}{k}\right) \cdot \left(x - \frac{1}{k+1}\right) - \frac{1}{k} = (x-\alpha_k)(x-\beta_k)(x-\gamma_k).$$

Observe that

$$\begin{aligned}
-P(0) &= -(0-\alpha_k)\cdot(0-\beta_k)\cdot(0-\gamma_k) = \alpha_k\beta_k\gamma_k \\
-P(0) &= -\left(\frac{-1}{k-1}\cdot\frac{-1}{k}\cdot\frac{-1}{k+1} - \frac{1}{k}\right) = \frac{1}{k\cdot(k^2-1)} - \frac{1}{k} = \frac{k}{k^2-1} \\
-P(-1) &= -(-1-\alpha_k)\cdot(-1-\beta_k)\cdot(-1-\gamma_k) = (1+\alpha_k)(1+\beta_k)(1+\gamma_k) \\
-P(-1) &= -\left(-1-\frac{1}{k-1}\right)\cdot\left(-1-\frac{1}{k}\right)\cdot\left(-1-\frac{1}{k+1}\right) - \frac{1}{k} = \frac{k^2+3k-1}{k\cdot(k-1)}.
\end{aligned}$$

Hence, we would like to compute

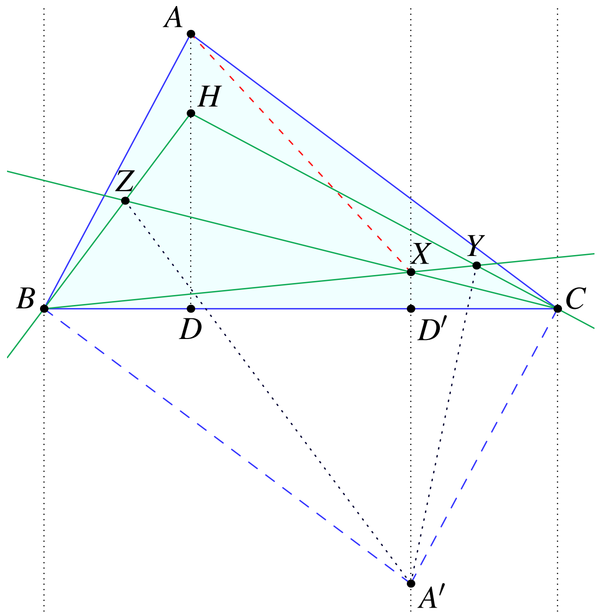
$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{\alpha_k\beta_k\gamma_k\cdot(1+\alpha_k)\cdot(1+\beta_k)\cdot(1+\gamma_k)}{k+1} &= \sum_{k=2}^{\infty} \frac{k^2+3k-1}{k\cdot(k-1)\cdot(k+1)}\cdot\frac{k}{k^2-1} \\
&= \sum_{k=2}^{\infty} \frac{k^2+3k-1}{(k-1)^2\cdot(k+1)^2} \\
&= \sum_{k=2}^{\infty} \left(\left(\frac{1/2}{k-1} - \frac{1/2}{k+1}\right) + \left(\frac{3/4}{(k-1)^2} - \frac{3/4}{(k+1)^2}\right)\right) \\
&= \left(1+\frac{1}{2}\right) + \frac{3}{4}\cdot\left(\frac{1}{1^2}+\frac{1}{2^2}\right) = \boxed{\frac{27}{16}}.
\end{aligned}$$

For the last two equations, we have applied partial fraction decomposition and telescoping sums.

Problem 9. Let ABC be a triangle with orthocenter H and $AB = 17, BC = 28, CA = 25$. Let X be a point whose distance to $\overline{BC}$ is 2. Suppose $\overline{BX}$ and $\overline{HC}$ intersects at Y, and $\overline{CX}$ and $\overline{HB}$ intersects at Z, such that $YZ < BC$ and $YC \cdot CA = ZB \cdot BA$. Find AX.

Proposed by Brian Yang

Solution: $\boxed{\sqrt{313}}$.



Let A' be the reflection of A over the midpoint of $\overline{BC}$, and let D, D' be the feet of the altitudes from A, A', respectively, onto $\overline{BC}$. Since $YC \cdot CA = ZB \cdot BA$, we have $\frac{A'C}{CY} = \frac{A'B}{BZ}$. Furthermore, from $\overline{AB} \parallel \overline{CA'}, \overline{AC} \parallel \overline{BA'}$ follows $\overline{A'C} \perp \overline{CY}, \overline{A'B} \perp \overline{BZ}$. Thus, $\triangle A'CY \sim \triangle A'BZ$. Assume on the contrary that this similarity is orientation preserving, i.e. $\measuredangle YA'C = \measuredangle ZA'B$, then we have $\triangle CA'B \sim \triangle YA'Z$ (orientation preserving). Note $A'Y \geq A'C, A'Z \geq A'B$, as $A'C, A'B$ are the altitudes of A' onto $\overline{HC}$ and $\overline{HB}$, respectively. Hence, $YZ \geq BC$, a contradiction. Thus, $\triangle A'CY \sim \triangle A'BZ$ is an orientation reversing similarity, i.e. $\measuredangle YA'C = \measuredangle BA'Z$. Let l_B, l_C be the perpendiculars to $\overline{BC}$ at B, C, respectively. By the converse to Jacobi's theorem on triangle $A'BC$, $\overline{A'X}$ intersects the point at infinity $l_B \cap l_C$, i.e. $X \in \overline{A'D'}$.

Since the distance from X to $\overline{BC}$ is defined, this leaves us with 2 possible choices for X: either $\overline{XD'}$ meets the interior of triangle ABC or it does not. Clearly $YZ < BC$ only in the former case. Now, observe that triangle BDA is an 8-15-17 right triangle, and triangle ADC is a 3-4-5 right triangle. Therefore, $AD = 15, BD = 8, DD' = 12, D'C = 8$. By the Pythagorean theorem, $AX = \sqrt{12^2 + (15-2)^2} = \boxed{\sqrt{313}}$.

Problem 10. Brian and Stephanie are sitting next to each other at a round table with a number of other people (possibly 0 other people). The people at the table pass a rubber ball to each other, always to the person on their left. The ball starts with Stephanie, and arrives at Brian after exactly 2024 passes.

Suppose that after k more passes ($1 \leq k \leq 2024$), Stephanie receives the ball again. How many possible values of k are there?

Proposed by Ritvik Teegavarapu

Solution: $\boxed{1153}$.

Let n be the number of people at the table. Label the people at the table $0, 1, 2, \ldots, n-1 \pmod{n}$ clockwise, such that Stephanie is labelled 0 (thus, Brian is either labelled $+1$ or -1). We say that an integer $1 \leq k \leq 2024$ is *good* if it satisfies the conditions of the problem statement. Since Stephanie receives the ball after $k + 2024$ passes, we have $n \mid k + 2024$.

Assume Brian is labeled by -1. Then, since Brian receives the ball after 2024 passes, we have $2024 \equiv -1 \pmod{n}$. Since $n \geq 2$, this means $\gcd(k + 2024, 2025) > 1$ is necessary. On the other hand, if $1 \leq k \leq 2024$ is such that $\gcd(k + 2024, 2025) > 1$, then k is certainly good, simply by taking $n = \gcd(k + 2024, 2025)$. Hence, in the Brian labelled by -1 case, k is good if and only if $\gcd(k+2024, 2025) = \gcd(k-1, 2025) > 1$. An analogous argument shows that in the Brian is labeled by $+1$ case, k is good if and only if $\gcd(k+2024, 2023) = \gcd(k+1, 2023) > 1$.

Thus, the task is to find the number of positive integers $1 \leq k \leq 2024$ such that $\gcd(k-1, 2025)$ or $\gcd(k+1, 2023)$ is greater than 1. Towards complementary counting, we shall compute the number of positive integers $0 \leq k \leq 2023$ such that $\gcd(k, 2025) = \gcd(k+2, 2023) = 1$ simultaneously. Noting the prime factorizations $2025 = 3^2 \cdot 5^2, 2023 = 7 \cdot 17^2$, we observe

$$\gcd(k, 2025), \gcd(k+2, 2023) = 1 \iff 3 \nmid k,\ 5 \nmid k,\ 7 \nmid k+2,\ 17 \nmid k+2$$

for any positive integer k. Thus, the Chinese Remainder theorem implies that among the positive integers $0 \leq k < 3 \cdot 5 \cdot 7 \cdot 17 = 1785$, there are exactly $\varphi(1785) = 2 \cdot 4 \cdot 6 \cdot 16 = 768$ positive integers such that $3 \nmid k,\ 5 \nmid k,\ 7 \nmid k+2,\ 17 \nmid k+2$. It remains to find the number of integers $1785 \leq k \leq 2023$ such that the above condition holds. It suffices to work over the range $0 \leq k \leq 2023 - 1785 = 238$.

In the range $0 \leq k < 210$, there are $2 \cdot \varphi(105) = 2 \cdot (2 \cdot 4 \cdot 6) = 96$ integers k such that $3 \nmid k,\ 5 \nmid k,\ 7 \nmid k+2$. Among these 96 integers, we check directly that $k = 32, 83, 134, 151, 202$ satisfies $17 \mid k+2$, and the other 91 of them satisfies $17 \nmid k+2$. In the range $210 \leq k < 239$, there are 15 integers k such that $3 \nmid k, 5 \nmid k$. Among these 15 integers k, we check directly that $k = 219, 229, 236$ satisfies either $7 \mid k+2$ or $17 \mid k+2$, so the other 12 of them satisfy $7 \nmid k+2, 17 \nmid k+2$.

Therefore, there are exactly $768 + 91 + 12 = 871$ positive integers $0 \leq k \leq 2023$ such that $\gcd(k, 2025), \gcd(k+2, 2023) = 1$. In other words, the number of good numbers $1 \leq k \leq 2024$ is just $2024 - 871 = \boxed{1153}$.

Problem 11. Pick a point P uniformly at random from the interior of an equilateral triangle ABC. What is the probability that the lengths PA, PB, PC determine a non-degenerate triangle of area at least $\frac{2}{9}$ that of triangle ABC?

Proposed by Brian Yang

Solution: $\boxed{\dfrac{9 + 2\pi\sqrt{3}}{27}}$.

We recall the following useful lemma due to Leibniz:

Lemma 1. Let ABC be a triangle, G its centroid. For any point P, we have

$$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3(GP^2).$$

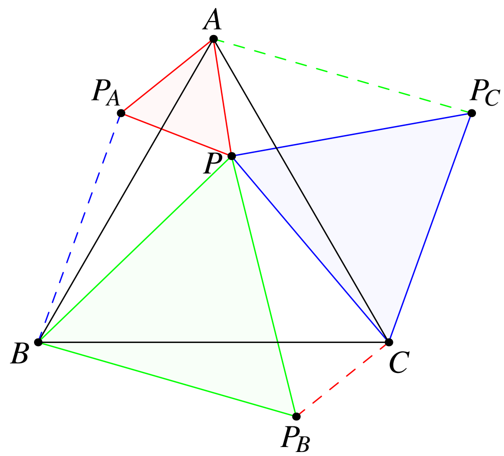
Proof. We can interpret everything here as vectors: let $\vec{GA} = \vec{x}, \vec{GB} = \vec{y}, \vec{GC} = \vec{z}, \vec{GP} = \vec{v}$. Then, by basic properties of the dot product:

$$\begin{aligned} PA^2 + PB^2 + PC^2 &= |\vec{x} - \vec{v}|^2 + |\vec{y} - \vec{v}|^2 + |\vec{z} - \vec{v}|^2 \\ &= (\vec{x} - \vec{v}) \cdot (\vec{x} - \vec{v}) + (\vec{y} - \vec{v}) \cdot (\vec{y} - \vec{v}) + (\vec{z} - \vec{v}) \cdot (\vec{z} - \vec{v}) \\ &= |\vec{x}|^2 + |\vec{y}|^2 + |\vec{z}|^2 + 3|\vec{v}|^2, \end{aligned}$$

which is exactly what we need to show. The last equality here is because $\vec{x} + \vec{y} + \vec{z} = \vec{0}$ by definition of centroid. □

Return to the original problem. WLOG we assume ABC is a unit equilateral triangle. Let Δ be the triangle determined by sides PA, PB, PC (we shall show Δ always exists), and $[\Delta]$ its area. In light of the above lemma, we shall see that $[\Delta]$ is determined by the length of $\overline{GP}$, where G is the centroid of the equilateral triangle ABC

(also the incenter, circumcenter of ABC). Assuming ABC is oriented counterclockwise, let P_A, P_B, P_C be the images of P by the 60° clockwise rotation with centers A, B, C, respectively:

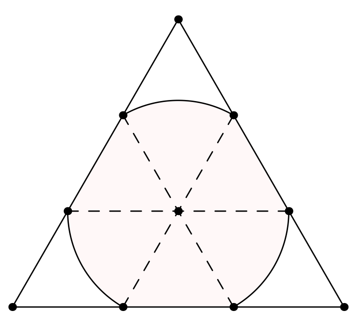


By rotation, $\triangle PBC \cong \triangle P_C AC, \triangle PCA \cong \triangle P_A BA, \triangle PAB \cong P_B CB$. Since $[PBC]+[PCA]+[PAB]=[ABC]$, we deduce $[AP_A BP_B CP_C] = 2\cdot[ABC]$ (as usual, $[\cdot]$ denotes area). Furthermore, the triangles PAP_A, PBP_B, PCP_C are all equilateral triangles (e.g., because $PA = PP_A$ and $\angle APP_A = 60^\circ$), though not necessarily mutually congruent, while the (non-degenerate) triangles $PP_A B, P_B CP, APP_C$ have the same side lengths as Δ, hence are mutually congruent with Δ. Applying the above lemma,

$$\begin{aligned}
[\Delta] &= \frac{1}{3}\left(2\cdot[ABC]-[PAP_A]-[PBP_B]-[PCP_C]\right)\\
&= \frac{1}{3}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\left(PA^2+PB^2+PC^2\right)\right)\\
&= \frac{1}{3}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\left(GA^2+GB^2+GC^2+3(GP^2)\right)\right) = \frac{\sqrt{3}}{12}-\frac{\sqrt{3}}{4}\cdot GP^2,
\end{aligned}$$

where we have used the fact $GA^2+GB^2+GC^2=1$.

Since $[ABC]=\frac{\sqrt{3}}{4}$, the area $[\Delta]$ is at least $\frac{2}{9}\cdot[ABC]=\frac{\sqrt{3}}{18}$ precisely when $GP^2 \le \frac{1}{9}$, i.e. $GP \le \frac{1}{3}$. Thus, the answer to the question is given precisely by the area of the shaded region divided by the area of triangle ABC in the following figure:



The shaded region is bounded by ABC and the circle with center G and radius $\frac{1}{3}$. This region consists of three equilateral triangles, each of side length $\frac{1}{3}$, and three circular arcs, each of measure 60° and radius $\frac{1}{3}$. Thus, the answer is

$$\frac{3\cdot\frac{\sqrt{3}}{4}\cdot(\frac{1}{3})^2+3\cdot\pi(\frac{1}{3})^2\cdot\frac{1}{6}}{\frac{\sqrt{3}}{4}}=\boxed{\frac{9+2\pi\sqrt{3}}{27}}.$$

Problem 12. Compute

$$\sum_{k=0}^{103}\left(\lfloor k(3-\sqrt{3})\rfloor-\left\lfloor\left\lfloor(2-\sqrt{3})(k+1)\right\rfloor\cdot(3+\sqrt{3})\right\rfloor\right)^2$$

(note that $\sqrt{3}\approx 1.73$).

Proposed by Ritvik Teegavarapu and Brian Yang

Solution: $\boxed{347}$.

For any real number $r>1$, introduce the set $\mathscr{B}_r:=\{\lfloor r\rfloor,\lfloor 2r\rfloor\dots\}$. Note $\mathscr{B}_r$ is a subset of the natural numbers $\mathbb{N}=\{1,2,\dots\}$. Recall:

Lemma 1 (Beatty's theorem). For any irrational numbers $r,s>1$ such that $\frac{1}{r}+\frac{1}{s}=1$, the sets $\mathscr{B}_r,\mathscr{B}_s$ partition $\mathbb{N}$.

Proof. Assume on the contrary there are integers $j,k,l>0$ such that $j=\lfloor kr\rfloor=\lfloor ls\rfloor$, i.e. $j\le kr,ls<j+1$. By irrationality of r,s, and j being non-zero, the inequalities here are strict: $j<kr,ls<j+1$; that is:

$$\frac{j}{r}<k<\frac{j+1}{r}\qquad\frac{j}{s}<l<\frac{j+1}{s}.$$

Summing these inequalities together and using the hypothesis $\frac{1}{r}+\frac{1}{s}=1$, we obtain $j<k+l<j+1$, a contradiction since $k+l$ is an integer. This proves $\mathscr{B}_r\cap\mathscr{B}_s=\emptyset$. A similar argument proves $(\mathbb{N}\setminus\mathscr{B}_r)\cap(\mathbb{N}\setminus\mathscr{B}_s)=\emptyset$. $\square$

We also note the following. Suppose $1<r<2$ is irrational, then for any integer $k>0$, we have $\lfloor(k+1)r\rfloor-\lfloor kr\rfloor\in\{1,2\}$. Furthermore, $\lfloor(k+1)(r-1)\rfloor=\lfloor k(r-1)\rfloor+1$ precisely when $\lfloor(k+1)r\rfloor-\lfloor kr\rfloor=2$. This holds because $\lfloor(k+1)(r-1)\rfloor=\lfloor(k+1)r\rfloor-(k+1)$, $\lfloor k(r-1)\rfloor=\lfloor kr\rfloor-k$. In lieu of these observations, we may obtain the following "improvement" to Lemma 1:

Lemma 2. Let $r,s>1$ be irrational numbers such that $r<2$, $\frac{1}{r}+\frac{1}{s}=1$. Let S be the set of all $k\in\mathbb{N}$ such that $\lfloor(k+1)r\rfloor-\lfloor kr\rfloor=2$. Then, for any $k\in S$,

$$\lfloor\lfloor(k+1)(r-1)\rfloor\cdot s\rfloor=\lfloor kr\rfloor+1.$$

Proof. Write S as an increasing sequence $k_1,k_2,\dots$. We induct on n, where n is the index of this sequence. For every $1\le k<k_1$, we have $\lfloor(k+1)(r-1)\rfloor=\lfloor k(r-1)\rfloor$ by the above observation. Hence, $\lfloor(k+1)(r-1)\rfloor=\lfloor 1\cdot(r-1)\rfloor=0$, i.e. $\lfloor\lfloor(k+1)(r-1)\rfloor\cdot s\rfloor=0$. Moreover, notice $\lfloor(k_1+1)(r-1)\rfloor=\lfloor k_1(r-1)\rfloor+1=1$, and so by Lemma 1, the only possible value of the number $\lfloor\lfloor(k_1+1)(r-1)\rfloor\cdot s\rfloor$ is exactly $\lfloor k_1r\rfloor+1$. Now inductively assume that the desired statement is true for $k_1,k_2,\dots,k_n\in S$, and that $\lfloor(k_i+1)(r-1)\rfloor=i$ for $1\le i\le n$. For $k_n<k<k_{n+1}$, we have $\lfloor(k+1)(r-1)\rfloor=\lfloor k(r-1)\rfloor$. Thus, $\lfloor(k_{n+1}+1)(r-1)\rfloor=\lfloor k_{n+1}(r-1)\rfloor+1=\lfloor(k_n+1)(r-1)\rfloor+1=n+1$, and again by Lemma 1, the only possible value of the number $\lfloor\lfloor(k_{n+1}+1)(r-1)\rfloor\cdot s\rfloor$ is exactly $\lfloor k_{n+1}r\rfloor+1$ (by induction, we have already accounted for n smallest values $\lfloor k_1r\rfloor+1,\lfloor k_2r\rfloor+1,\dots\lfloor k_nr\rfloor+1\in\mathscr{B}_s$). $\square$

In the situation of the above Lemma 2, we may also observe the following: for integers $1\le k<\min S$, we have $\lfloor\lfloor(k+1)(r-1)\rfloor\cdot s\rfloor=0$. Otherwise, suppose $k\in\mathbb{N}\setminus S$. Then, if k' is the largest number in S such that $k'\le k$, then we have $\lfloor\lfloor(k'+1)(r-1)\rfloor\cdot s\rfloor=\lfloor\lfloor(k+1)(r-1)\rfloor\cdot s\rfloor$.

Turning to the original problem, set $r = 3-\sqrt{3}, s = 3+\sqrt{3}$, and notice $r-1 = 2-\sqrt{3}$ and $\frac{1}{r}+\frac{1}{s} = 1$. Here, we shall use the decimal approximation $\sqrt{3} \approx 1.73$. In lieu of the above two lemmas, we must understand the elements of $\mathscr{B}_s$ in the set of integers $1, 2, \cdots, 130$, as $130 = \lfloor 103(3-\sqrt{3})\rfloor$. We have $\lfloor 27(3+\sqrt{3})\rfloor = 127, \lfloor 28(3+\sqrt{3})\rfloor = 132$. For any non-negative integer k, there are either 3 or 4 elements of $\mathscr{B}_r$ strictly between $\lfloor k(3+\sqrt{3})\rfloor$ and $\lfloor (k+1)(3+\sqrt{3})\rfloor$, since $4 < 3+\sqrt{3} < 5$. Thus, the sequence $0, \lfloor (3+\sqrt{3})\rfloor, \cdots \lfloor 27(3+\sqrt{3})\rfloor \cdots \lfloor 28(3+\sqrt{3})\rfloor$ partitions $\mathscr{B}_r \cap \{1, 2, \cdots, 132\}$ into 28 classes in the obvious way. Since $|\mathscr{B}_r \cap \{1, 2, \cdots, 132\}| = 132-28 = 104$, the case of "3 elements between" (resp. "4 elements between") occurs for exactly 8 (resp. exactly 20) classes. By Lemma 2, and the above observation, a "3 elements between" (resp., a "3 elements between") class contributes the terms $1^2, 2^2, 1^2$ (resp. the terms $1^2, 2^2, 3^2, 1^2$) to the sum in the problem statement. Here, the terms are listed in the order corresponding to writing the elements of a class in increasing order. Hence,

$$\sum_{k=1}^{104}\left(\lfloor k(3-\sqrt{3})\rfloor - \left\lfloor \left\lfloor (2-\sqrt{3})(k+1)\right\rfloor \cdot (3+\sqrt{3})\right\rfloor\right)^2$$
$$= (1^2+2^2+1^2)\cdot 8 + (1^2+2^2+3^2+1^2)\cdot 20 = 348.$$

The desired sum is indexed on $k = 0, 1, \cdots, 103$. The term arising from $k = 104$ contributes 1^2 to the above sum, as $\lfloor 104(3-\sqrt{3})\rfloor + 1 = 132 = \lfloor 28(3+\sqrt{3})\rfloor$. The term arising from $k = 0$ contributes nothing. Thus, the answer is $\boxed{347}$.

Problem 13. Let N be the number of distinct tuples $(x_1, x_2, \ldots, x_{46})$ of positive integers with $x_1, x_2, \ldots, x_{46} \leq 88$ such that the remainder when $x_1^{35} + x_2^{35} + \cdots + x_{46}^{35}$ is divided by 2024 is 253. Compute the remainder when N is divided by 46.

Proposed by Justin Lee

Solution: $\boxed{40}$.

If a tuple $(x_1, \ldots, x_{46})$ is a solution to the above congruence, then so is $(x_i, \ldots, x_{45+i})$ $(1 \leq i \leq 46)$ where indices are taken modulo 46; that is, cyclic shifts of a solution are also solutions. Thus, we may form a partition on the set of solutions to the desired congruence, where two solutions lie in the same partition class if and only if one is a cyclic shift of the other. It suffices to find the number of solutions that have fewer than 46 distinct cyclic shifts.

Consider any 46-tuple $\underline{x} := (x_1, \ldots, x_{46})$. Let d be the smallest positive integer such that the cyclic shift $(x_{1+d}, \ldots, x_{46+d})$ of $\underline{x}$ equals $\underline{x}$. Then, for any cyclic shift $(x_{1+k}, \ldots, x_{46+k})$ (k an integer) with $(x_{1+k}, \ldots, x_{46+k}) = \underline{x}$, observe that k is necessarily a multiple of d. Thus, $d \mid 46$, and the number of distinct cyclic shifts of a $\underline{x}$ is $\frac{46}{d}$, a divisor of 46. If $\underline{x}$ has 1 or 23 distinct cyclic shifts, then we must have $x_i = x_{i+23}$ for all i. However, this would mean that $x_1^{35} + \cdots + x_{46}^{35}$ is even, so that $\underline{x}$ is not a solution of the above congruence. In the case where $\underline{x}$ has exactly 2 distinct cyclic shifts, we have that $x_1 = x_3 = \ldots = x_{45}$ and $x_2 = x_4 = \ldots = x_{46}$. Hence, the question reduces to counting the number of distinct pairs (a, b) of positive integers with $a, b \leq 88$ and $23(a^{35}+b^{35}) \equiv 253 \pmod{2024}$, which is equivalent to $a^{35}+b^{35} \equiv 11 \pmod{88}$. Note that any solution to this latter congruence has a, b distinct.

By the Chinese remainder theorem, the number of solutions to $a^{35}+b^{35} \equiv 11 \pmod{88}$ equals the product of the number of solutions of $a^{35}+b^{35} \equiv 3 \pmod 8$ $(0 < a, b \leq 8)$ and the number of solutions of $a^{35}+b^{35} \equiv 0 \pmod{11}$ $(0 < a, b \leq 11)$. First, we count solutions modulo 8. Recall that for an integer n coprime to 8, the multiplicative order modulo 8 is either 1 or 2, with the former case occurring precisely with $n \equiv 1 \pmod 8$. Thus, if $a^{35}+b^{35} \equiv 3 \pmod 8$, then one of a or b must be even, in which case the other must be congruent to 3 modulo 8, yielding $4 \cdot 2 = 8$ solutions.

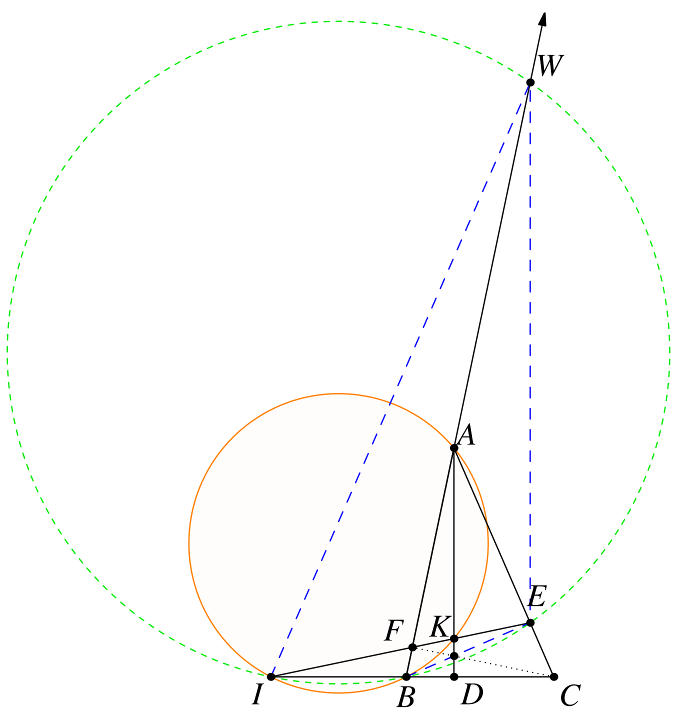
Now, we count solutions modulo 11. Recall that the congruence $n^2 \equiv 1 \pmod{11}$ has 2 solutions $\pmod{11}$:

$n = \pm 1$. Thus, given $a^{35} + b^{35} \equiv 0 \pmod{11}$ $(0 < a, b \leq 11)$, Fermat's little theorem shows a^{35} (and also b^{35}) can only be congruent to -1, 0, or 1: if $a \not\equiv 0 \pmod{11}$, then $(a^{35})^2 \equiv a^{70} \equiv 1 \pmod{11}$. Notice $n^{35} \equiv n^5 \equiv -1 \pmod{11}$ for exactly 5 of the non-zero residues $n \in \{1, 2, \ldots, 10\}$ modulo 11; an easy way to see this is the fact $n^5 \equiv \pm 1 \pmod{11}$ if and only if $(-n)^5 \equiv \mp 1 \pmod{11}$. The case where both a^{35}, b^{35} are congruent to 0 $\pmod{11}$ gives 1 solution, whereas the case where one of a^{35}, b^{35} is $1 \pmod{11}$ and the other is $-1 \pmod{11}$ gives $2 \cdot 5 \cdot 5 = 50$ solutions. So the modulo 11 case yields $50 + 1 = 51$ total solutions. All told, we obtain $8 \cdot 51 \equiv 408 \equiv \boxed{40} \pmod{46}$ solutions to the $\pmod{88}$ congruence.

Problem 14. Let ABC be an acute triangle with $AC > AB$. Let D, E, and F be the feet of the altitudes from A, B, and C, onto $\overline{BC}, \overline{CA}$ and $\overline{AB}$, respectively. Let K be the intersection of $\overline{EF}$ and $\overline{AD}$, and let I be the intersection of $\overline{EF}$ and $\overline{BC}$. Let W be a point on ray $\overrightarrow{BF}$ such that $\angle IWF = \angle FWE$. Suppose $AKBI$ is cyclic and $\cos(\angle BCA) = \frac{2}{5}$. Find $\frac{WB}{WI}$.

Proposed by August Chen

Solution: $\boxed{\dfrac{7\sqrt{15} - 3\sqrt{35}}{10}}$.



Let H be the orthocenter of ABC. Notice $AKBI$ cyclic implies

$$\measuredangle BAK = \measuredangle BIK = \measuredangle CBE - \measuredangle FEB = \measuredangle DAC - \measuredangle BAH,$$

i.e. $2B - C = 90^\circ$ (using the usual notation for the angles of triangle ABC). Furthermore:

Claim 1. The quadrilateral $WIBE$ is cyclic.

Proof 1. Let $W' = (EBI) \cap \overline{BF}$ where $W' \neq B$. Observe W is unique: it is the second intersection of $\overrightarrow{BF}$ with the Apollonius circle determined by segment $\overline{IE}$ and point F. Thus, it is enough to show $W' = W$. To this end, first note $\measuredangle BIE = \measuredangle BAH = \measuredangle FEB = \measuredangle IEB$, where the first equality comes from $AKBI$ cyclic. Then, $W'IBE$ cyclic implies $\measuredangle IW'F = \measuredangle FW'E$, so $W' = W$. □

From here, it follows by $\measuredangle WBI = \measuredangle WEI$ that $\triangle WIB \sim \triangle WFE$. We compute

$$\frac{WB}{WI} = \frac{WE}{WF} = \frac{IB}{IF} = \frac{\sin C}{\sin B} = \frac{\sin C}{\sin(45^\circ + C/2)} = \frac{2\sin C}{\sqrt{\cos C + 1} + \sqrt{-\cos C + 1}},$$

so the answer is

$$\frac{2\sqrt{21}}{5} \cdot \frac{1}{\sqrt{\frac{7}{5}} + \sqrt{\frac{3}{5}}} = \frac{2\sqrt{105}(\sqrt{7} - \sqrt{3})}{20} = \boxed{\frac{7\sqrt{15} - 3\sqrt{35}}{10}}.$$

Problem 15. A function $f : \{1,2,3,4,5,6,7\} \to \{2,3,4,5,6\}$ is called *roughly monotonic* if for every integer $2 \le k \le 5$ in which there is some integer $1 \le a \le 7$ with $f(a) = k$, then there exist integers $1 \le a_0 < b \le 7$ such that $f(a_0) = k, f(b) = k+1$. Compute the number of roughly monotonic functions f.

Proposed by Brian Yang

Solution: $\boxed{4919}$.

Let us consider roughly monotonic functions $f : [n] \to [n]$ ($[n] := \{1,2,\ldots,n\}$) "on n," where we let k range over $1 \le k \le n-1$. Write any function f as a sequence $f := (f(1), f(2), \ldots, f(n))$ The main claim is:

Claim 1. The set of all roughly monotonic functions $(f(1), f(2), \ldots, f(n))$ are in bijection with the set of all permutations $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ of $\{1,2,\ldots,n\}$.

Proof 1. We propose two mappings: one that constructs a roughly monotonic function f on n, given any permutation $\sigma := (\sigma(1), \sigma(2), \ldots, \sigma(n))$, and the other that builds a permutation σ of $[n]$, given any roughly monotonic function $f := (f(1), f(2), \ldots, f(n))$.

Given any permutation $\sigma := (\sigma(1), \sigma(2), \ldots, \sigma(n))$, construct f as follows. Initially, f is undetermined everywhere. At step 1, find $1 \le i_1 \le n$ such that $\sigma(i_1) = 1$, and set $f(i_1) = n$ (this determines f at one value, namely $i_1 = \sigma^{-1}(1)$). At step $1 < j \le n$, say we have determined f at the $j-1$ values $\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(j-1)$, i.e. the positions of $1,2,\ldots,j-1$ in σ. Let m be the value of f at $\sigma^{-1}(j-1)$. Then, if $i_j = \sigma^{-1}(j) > \sigma^{-1}(j-1)$, i.e. $j-1$ precedes j in σ, then we set $f(i_j) = m$. Otherwise, j precedes $j-1$ in σ, and we set $f(i_j) = m-1$. As an example, given the permutation $(7,2,4,1,5,6,3)$, we obtain the function $f := (4,6,5,7,5,5,6)$.

Note that for any $m \in [n]$, the pre-image $f^{-1}(m)$ describes the positions of an increasing subsequence of $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ consisting of consecutive integers (possibly empty). To be precise: $f^{-1}(n)$ are the positions of $1,2,\ldots,l$ in σ, for some $l \ge 1$. Moreover, if $m < n$ and $f^{-1}(m)$ is non-empty, then by construction, $f^{-1}(m)$ are the positions of some $1 < j, j+1, \ldots, j+l \le n$ for some $l \ge 0$ in σ. In particular, it is necessary that j precedes $j-1$ in σ, else $j-1 \in f^{-1}(m)$. As a result, $j-1 \in f^{-1}(m+1)$. This verifies f is indeed roughly monotonic.

Now, given any roughly monotonic function $f := (f(1), f(2), \ldots, f(n))$, construct σ as follows. Initially, σ is undetermined anywhere. At step 1, find the smallest $1 \le i_1 \le n$ such that $f(i)$ achieves the maximum of f, and set $\sigma(i_1) = 1$. At step $1 < j \le n$, assume that we have determined σ at the $j-1$ distinct values $i_1, i_2, \ldots, i_{j-1}$. Then, find the smallest $1 \le i_j \le n$ such that $i_j \notin \{i_1, i_2, \ldots, i_{j-1}\}$, and $f(i_j)$ achieves the maximum of the restricted function

$$f|_{[n]\setminus\{i_1,i_2,\ldots i_{j-1}\}} : [n] \setminus \{i_1, i_2, \ldots i_{j-1}\} \to [n].$$

Set $\sigma(i_j) = j$. It is clear that σ is a permutation.

Now, we check that the two mappings $\sigma \mapsto f, f \mapsto \sigma$ are inverse of each other. First, given a permutation σ, note that we construct f at step j by determining its value at $i_j := \sigma^{-1}(j)$, for each $1 \le j \le n$. However, in this construction of f, observe that i_j is indeed the smallest value such that $i_j \notin \{i_1, i_2, \ldots i_{j-1}\}$, and $f(i_j)$ achieves the maximum of the restriction $f|_{[n]\setminus\{i_1,i_2,\ldots i_{j-1}\}}$. That is, the two ways we defined the i_j's are the same, in the sense that these definitions are "preserved" by both of the above mappings. Since $\sigma^{-1}(j) = i_j$, the composition of $\sigma \mapsto f$ with $f \mapsto \sigma$ is precisely the identity.

On the other hand, given a roughly monotonic function f, say that we construct a permutation σ from f, and then a roughly monotonic function f from σ; we must check $f = f'$. Once again, we record the values $i_j := \sigma^{-1}(j),\ 1 \leq j \leq n$. Then, each i_j is the smallest value satisfying the achieving the maximum of the restriction condition for *both functions* f and f'! Since $f(i_1)$ (resp. $f'(i_1)$) is the maximum of f (resp. f'), we must have $f(i_1) = n$ (resp. $f'(i_1) = n$), else f (resp. f') cannot be roughly monotonic. Assume inductively that we have equality $f(i_{j-1}) = f'(i_{j-1})$ for some $j \geq 2$. If $i_j > i_{j-1}$, then the achieving the maximum of restriction condition forces $f(i_j) \leq f(i_{j-1}), f'(i_j) \leq f'(i_{j-1})$, but the roughly monotonic condition implies that we must have $f(i_j) = f'(i_j) = f(i_{j-1}) = f'(i_{j-1})$. Otherwise, if $i_j < i_{j-1}$, then the achieving the maximum of restriction condition forces $f(i_j) < f(i_{j-1}), f'(i_j) < f'(i_{j-1})$, but then the roughly monotonic condition forces $f(i_j) = f'(i_j) = f(i_{j-1}) - 1 = f'(i_{j-1}) - 1$. This verifies that the composition of $f \mapsto \sigma$ with $\sigma \mapsto f$ is the identity. □

In light of the above claim, and in the problem statement, the restriction of the codomain to $\{2, 3, \ldots, 6\}$ implies we must compute the number of roughly monotonic functions $f : [7] \to [7]$ such that the range of f is of size at most 5. Towards complementary counting, we must count permutations of [7] that split into 6 or 7 increasing subsequences (in concomitance with the definition of the mapping $\sigma \mapsto f$). In the case of 7 increasing subsequences, each increasing subsequence is a singleton, and so we just get one permutation $\sigma = (7, 6, 5, 4, 3, 2, 1)$. In the case of 6 increasing subsequences, one of the increasing subsequences is of length 2, all of the others are length 1. There are 6 possible cases for the length 2 subsequence in σ: $(1, 2), (2, 3), \ldots, (6, 7)$. We may do casework over the possible length 2 subsequences in σ:

- $(1, 2)$: The collection $7, 6, 5, 4, 3$ are permuted in that order by σ. Hence, we must have $i_2 = 7$, and any $1 \leq i_1 \leq 6$ is sufficient. This yields $\binom{6}{1} = 6$ permutations.
- $(2, 3)$: Here, the collection $7, 6, 5, 4$ are permuted in that order by σ. Thus, it is necessary $i_3 \geq 6$, obtaining two (sub)-cases: The first case is that $i_3 = 7$, in which $1 \leq i_2 < i_1 \leq 6$ is necessary and sufficient. In the second case, we have $i_3 = 6$, forcing $i_1 = 7$, in which case any $1 \leq i_2 \leq 5$ is sufficient. This yields $\binom{6}{2} + \binom{5}{1} = 20$ permutations.
- $(3, 4)$: Here, the two collections $7, 6, 5$ and $2, 1$ are permuted in that order. Thus, it is necessary $i_4 \geq 5$, obtaining three (sub)-cases: The first case is that $i_4 = 7$, in which $1 \leq i_3 < i_2 < i_1 \leq 6$ is necessary and sufficient. In the second case, we have $i_4 = 6$, forcing $i_1 = 7$, in which case any $1 \leq i_3 < i_2 \leq 5$ is sufficient. In the third case, we have $i_4 = 5$, forcing $i_2 = 6, i_1 = 7$, in which case any $1 \leq i_3 \leq 4$ is sufficient. This yields $\binom{6}{3} + \binom{5}{2} + \binom{4}{1} = 34$ permutations.

Now, observing symmetry, the cases $(4, 5), (5, 6), (6, 7)$ yield the same number of permutations as the cases $(1, 2), (2, 3), (3, 4)$, respectively. There are $7! = 5040$ permutations in total, so the number of roughly monotonic functions requested by the problem statement is $5040 - 1 - 2(6 + 20 + 34) = \boxed{4919}$.