CMM 2024 Integration Bee Final Round Solutions

Problem 1. $\displaystyle\int \frac{\sec^2(x)}{\tan^2(x)-3\tan(x)+2}\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{\ln\left|\frac{\tan(x)-2}{\tan(x)-1}\right|+C}$

A u-substitution in the form of $u=\tan(x)$ is readily apparent since $\mathrm{du}=\sec^2(x)\,\mathrm{dx}$ is the numerator of our integrand. As such, we make the substitution and factor the denominator, yielding the following equivalent form of the integral.

$$\int \frac{\sec^2(x)}{\tan^2(x)-3\tan(x)+2}\,\mathrm{dx}=\int \frac{1}{u^2-3u+2}\,\mathrm{du}=\int \frac{1}{(u-2)(u-1)}\,\mathrm{du}.$$

Then, partial fraction decomposition results in the following integration.

$$\int \frac{1}{(u-2)(u-1)}\,\mathrm{du}=\int\left(\frac{1}{u-2}-\frac{1}{u-1}\right)\mathrm{du}=\ln|u-2|-\ln|u-1|+C.$$

Utilizing properties of logarithms and reintroducing $u=\tan(x)$ allows

$$\ln|u-2|-\ln|u-1|+C=\ln\left|\frac{u-2}{u-1}\right|+C=\boxed{\ln\left|\frac{\tan(x)-2}{\tan(x)-1}\right|+C}.$$

Problem 2. $\displaystyle\int_{\frac{2}{\pi}}^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{1}$

We notice $u=\frac{1}{x} \implies \mathrm{du}=-\frac{1}{x^2}\,\mathrm{dx}$ might lend itself well. We also notice that if $x=\frac{2}{\pi}$, $u=\frac{\pi}{2}$, and as $x\to\infty$, $u\to 0$, which give us the new bounds of our integral with respect to u. Evaluating accordingly, we have

$$\int_{\frac{2}{\pi}}^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{dx}=\lim_{b\to\infty}\int_{\frac{2}{\pi}}^{b} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{dx}=\int_{\frac{\pi}{2}}^{0} -\sin(u)\,\mathrm{du}=\cos(u)\Big|_{\frac{\pi}{2}}^{0}=\cos(0)-\cos\left(\frac{\pi}{2}\right)=\boxed{1}.$$

Problem 3. $\displaystyle\int \frac{1}{x\ln(x)\ln(\ln(x))}\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{\ln|\ln(\ln(x))|+C}$

An immediately intuitive way of solving this integral comes from noticing the repeated chain rule resultants from the natural logarithm. A more concise solution begins with the substitution $u=\ln(\ln(x))$, which implies $\mathrm{du}=\frac{1}{x\ln(x)}\,\mathrm{dx}$. Thus,

$$\int \frac{1}{x\ln(x)\ln(\ln(x))}\,\mathrm{dx}=\int \frac{1}{u}\,\mathrm{du}=\ln|u|+C=\boxed{\ln|\ln(\ln(x))|+C}.$$

Problem 4. $\int -\frac{x \arccos(x)}{\sqrt{1-x^2}}\,\text{dx}$

Proposed by Sam Murray

Solution: $\boxed{\sqrt{1-x^2}\arccos(x)+x+C}$

We notice an implication of integration by parts in the form of $a = \arccos(x)$ and $\text{db} = -\frac{x}{\sqrt{1-x^2}}\,\text{dx}$. We solve for b explicitly as follows.

$$b = \int \text{db} = \int -\frac{x}{\sqrt{1-x^2}}\,\text{dx} = \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,\text{dx};$$

With $u = 1-x^2$ and $\text{du} = -2x\,\text{dx}$,

$$\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,\text{dx} = \frac{1}{2}\int u^{-\frac{1}{2}}\,\text{du} = \sqrt{u} = \sqrt{1-x^2}.$$

In this case, it is acceptable to ignore the constant of integration because of its eventual absorption. So,

$$\int a\,\text{db} = ab - \int b\,\text{da} = \sqrt{1-x^2}\arccos(x) - \int -\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\,\text{dx} = \sqrt{1-x^2}\arccos(x) + \int\,\text{dx}$$

$$\sqrt{1-x^2}\arccos(x) + \int\,\text{dx} = \boxed{\sqrt{1-x^2}\arccos(x)+x+C}.$$

Problem 5. $\int_1^{2e} \lfloor \ln(x) \rfloor\,\text{dx}$

Proposed by Sam Murray

Solution: $\boxed{e}$

Here, we want to separate our integration into multiple intervals. We notice that for $x \in [1,e)$, $\lfloor \ln(x) \rfloor = 0$. We also notice that for $x \in [e,e^2)$, $\lfloor \ln(x) \rfloor = 1$; but recall that our upper bound is $2e$ for our integral, necessitating that we find $2e < e^2$ to be true. One way to show this is that

$$2e < e^2 \iff 2 < e,$$

which we know to be true since $e \approx 2.7$. Thus, we can make the equivalence

$$\int_1^{2e} \lfloor \ln(x) \rfloor\,\text{dx} = \int_1^e 0\,\text{dx} + \int_e^{2e} 1\,\text{dx} = 2e - e = \boxed{e}.$$

Problem 6. $\int_0^{2\sqrt{506}} \sqrt{2024-x^2}\,\text{dx}$

Proposed by Sam Murray

Solution: $\boxed{506\pi}$

While possible, we avoid solving this using trigonometric substitution; instead we let our integrand equal y such that

$$y = \sqrt{2024 - x^2} \implies x^2 + y^2 = 2024,$$

where $y \geq 0$. This means that the integrand can be represented implicitly as a semicircle centered at the origin with radius $\sqrt{2024} = 2\sqrt{506}$, which is the upper bound of our integrand. Since our lower bound is the origin, we integrate from the center of such semicircle to its rightmost extremity, $2\sqrt{506}$, implying that our integral is precisely equal to a quarter-circle with radius $\sqrt{2024}$. Thus,

$$\int_0^{2\sqrt{506}} \sqrt{2024 - x^2}\ \mathrm{dx} = \frac{(\sqrt{2024})^2 \pi}{4} = \boxed{506\pi}.$$

Problem 7. $\displaystyle\int_{-e}^{e} \frac{e^{-x^{2024}} \cos(2024x)}{\arctan(2024x)}\ \mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{0}$

When integrating an odd integrand $o(x)$ across symmetric bounds, we have that

$$\int_{-a}^{a} o(x)\ \mathrm{dx} = 0.$$

In the case of this integrand, we see that it is indeed odd since both $e^{-x^{2024}}$ and $\cos(2024x)$ are even, and $\arctan(2024x)$ is odd, meaning that their quotient as shown in the integrand is odd, as well. Accordingly,

$$\int_{-e}^{e} \frac{e^{-x^{2024}} \cos(2024x)}{\arctan(2024x)}\ \mathrm{dx} = \boxed{0}.$$

Problem 8. $\displaystyle\int \frac{x \sin(x)}{\cos^2(x)}\ \mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{x\sec(x) - \ln|\sec(x) + \tan(x)| + C}$

Rewriting the integrand such that

$$\int \frac{x \sin(x)}{\cos^2(x)}\ \mathrm{dx} = \int x \sec(x) \tan(x)\ \mathrm{dx}$$

motivates an integration by parts with $a = x$ and $\mathrm{db} = \sec(x)\tan(x)\ \mathrm{dx}$. Then,

$$\int x \sec(x) \tan(x)\ \mathrm{dx} = \int a\ \mathrm{db} = ab - \int b\ \mathrm{da} = x\sec(x) - \int \sec(x)\ \mathrm{dx}.$$

The integral of secant is well-known, so, for brevity, we omit the process of evaluation, yielding that

$$x\sec(x) - \int \sec(x)\ \mathrm{dx} = \boxed{x\sec(x) - \ln|\sec(x) + \tan(x)| + C}.$$

Problem 9. $\displaystyle\int \frac{9x+18}{5\sqrt[5]{x+2}}\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{(x+2)^{\frac{9}{5}}+C}$

A u-substitution with $u=\sqrt[5]{x+2}$ will prove useful. As such, we find $x=u^5-2$ and thus $\mathrm{dx}=5u^4\,\mathrm{du}$. Cumulatively,

$$\int \frac{9x+18}{5\sqrt[5]{x+2}}\,\mathrm{dx}=\int \frac{(9(u^5-2)+18)(5u^4)}{5u}\,\mathrm{du}=\int \frac{(9u^5)(5u^4)}{5u}\,\mathrm{du}=\int 9u^8\,\mathrm{du}=u^9+C.$$

Reversing our substitution with $u=\sqrt[5]{x+2}$, we gain our final answer:

$$\boxed{(x+2)^{\frac{9}{5}}+C}.$$

Problem 10. $\displaystyle\int_0^4 \{x\}^4\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{\frac{4}{5}}$

The fractional part of x, $\{x\}$, is defined such that $\{x\}=x-\lfloor x\rfloor$. Therefore,

$$\int_0^4 \{x\}^4\,\mathrm{dx}=\int_0^4 (x-\lfloor x\rfloor)^4\,\mathrm{dx}.$$

By the definition of the floor function, for $n\in\mathbb{Z}$ and for $x\in[n,n+1)$, $\lfloor x\rfloor=n$. To ensure we cover our region of integration utilizing this definition, we need to construct four separate integrals with $n=0$, $n=1$, $n=2$, and $n=3$, all of which will be summed together. We mathematically interpret this necessity as

$$\sum_{n=0}^{3}\int_n^{n+1}(x-n)^4\,\mathrm{dx}.$$

We now proceed in integrating $(x-n)^4$ from n to $n+1$.

$$\int_n^{n+1}(x-n)^4\,\mathrm{dx}=\left.\frac{(x-n)^5}{5}\right|_n^{n+1}=\frac{1}{5}.$$

This means that our integral resultant is simply

$$\sum_{n=0}^{3}\frac{1}{5}=4\left(\frac{1}{5}\right)=\boxed{\frac{4}{5}}.$$

Problem 11. $\displaystyle\int \frac{1}{\sqrt{\sqrt{x}+2}}\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{\frac{4(\sqrt{x}+2)^{\frac{3}{2}}}{3}-8\sqrt{\sqrt{x}+2}+C}$

In order to construct a more useful integrand, we let $u=\sqrt{\sqrt{x}+2}$. Then, $x=(u^2-2)^2$, implying that $\mathrm{dx}=4u(u^2-2)\ \mathrm{du}$. Making our substitution, we have that

$$\int \frac{1}{\sqrt{\sqrt{x}+2}}\ \mathrm{dx}=\int 4(u^2-2)\ \mathrm{du}=4\left(\frac{u^3}{3}-2u\right)+C=\frac{4u^3}{3}-8u+C.$$

Finally, we obtain an expression in terms of x in that

$$\frac{4u^3}{3}-8u+C=\boxed{\frac{4(\sqrt{x}+2)^{\frac{3}{2}}}{3}-8\sqrt{\sqrt{x}+2}+C}.$$

Problem 12. $\int x^9\ln(x)\ \mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{\frac{x^{10}\ln(x)}{10}-\frac{x^{10}}{100}+C}$

We begin in designating $u=\ln(x)$; in turn, $x=e^u$ and therefore $\mathrm{dx}=e^u\ \mathrm{du}$. With this, we see that

$$\int x^9\ln(x)\ \mathrm{dx}=\int ue^{10u}\ \mathrm{du}.$$

Integration by parts easily results in

$$\int ue^{10u}\ \mathrm{du}=\frac{ue^{10u}}{10}-\int\frac{e^{10u}}{10}\ \mathrm{du}=\frac{ue^{10u}}{10}-\frac{e^{10u}}{100}+C.$$

After reversing our substitution and simplifying, we have that

$$\frac{ue^{10u}}{10}-\frac{e^{10u}}{100}+C=\boxed{\frac{x^{10}\ln(x)}{10}-\frac{x^{10}}{100}+C}.$$

Problem 13. $\int_0^9\frac{\sqrt{x}}{9+x}\ \mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{6-\frac{3\pi}{2}}$

To remove the $\sqrt{x}$ term from the numerator of the integrand, we let $u^2=x$; then, $2u\ \mathrm{du}=\mathrm{dx}$. We further notice that when $x=0$, $u=0$, and when $x=9$, $u=3$ since the co-domain of $\sqrt{x}$ is non-negative. With this in mind, we move to make our substitution, yielding

$$\int_0^9\frac{\sqrt{x}}{9+x}\ \mathrm{dx}=\int_0^3\frac{u2u}{9+u^2}\ \mathrm{du}=2\int_0^3\frac{u^2}{u^2+9}\ \mathrm{du}.$$

We now manipulate the integrand by adding 0 to it in a special form, as follows.

$$2\int_0^3 \frac{u^2}{u^2+9}\,\text{du} = 2\int_0^3 \frac{u^2+9-9}{u^2+9}\,\text{du} = 2\left(\int_0^3 \text{du} - 9\int_0^3 \frac{1}{u^2+9}\,\text{du}\right) = 6 - 18\int_0^3 \frac{1}{u^2+9}\,\text{du}.$$

Now, the integrand simply results in an arctangent form; specifically,

$$6 - 18\left[\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]_0^3 = 6 - 6\left[\arctan\left(\frac{u}{3}\right)\right]_0^3 = 6 - 6\left(\frac{\pi}{4}\right) = \boxed{6 - \frac{3\pi}{2}}.$$

Problem 14. $\displaystyle\int \frac{1}{2+e^x+2e^{-x}}\text{dx}$

Proposed by Sam Murray

Solution: $\boxed{\arctan(e^x+1)+C}$

Multiplying by 1 in the form of e^x/e^x, we see that

$$\int \frac{1}{2+e^x+2e^{-x}}\text{dx} = \int \frac{e^x}{e^{2x}+2e^x+2}\text{dx}.$$

The denominator can then be transformed into a sum of squares utilizing $2 = 1+1$, as shown.

$$\int \frac{e^x}{e^{2x}+2e^x+2}\text{dx} = \int \frac{e^x}{(e^x+1)^2+1}\text{dx}.$$

Then, a choice of $u = e^x$ such that $\text{du} = e^x\text{dx}$ is apparent; thus

$$\int \frac{e^x}{(e^x+1)^2+1}\text{dx} = \int \frac{1}{u^2+1}\text{du},$$

which results in the arctangent and therefore our solution:

$$\int \frac{1}{u^2+1}\text{du} = \arctan(u)+C = \boxed{\arctan(e^x+1)+C}.$$

Problem 15. $\displaystyle\int_0^\infty \frac{1}{(1+x^2)(1+2024^{\ln x})}\,\text{dx}$

Proposed by Katia Avanesov

Solution: $\boxed{\frac{\pi}{4}}$

Let I be the integral. We can exploit a symmetry in the integral by letting $u = \frac{1}{x}$ such that $du = -\frac{1}{x^2}\text{dx}$ to obtain

$$I = \int_0^\infty \frac{1}{(1+u^2)(1+2024^{\ln\frac{1}{u}})}\text{dx}$$

Since $\ln\frac{1}{u} = -\ln u$, we can add our two expressions for I to obtain

$$2I = \int_0^\infty \frac{1}{1+x^2}\left[\frac{1}{1+2024^{\ln x}} + \frac{1}{1+2024^{-\ln x}}\right]\text{dx}$$

Where we see that the term in square brackets simplifies to 1, so that $2I$ is just equal to the antiderivative of $\arctan(x)$. Hence, upon evaluating the bounds, we obtain $2I = \frac{\pi}{2}$, so $\boxed{I = \frac{\pi}{4}}$.

Problem 16. $\int_{-1}^{0} x\sqrt{1+x}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{-\frac{4}{15}}$

To eliminate the expression inside the square root, we simply consider $u = 1+x$, which implies that $\mathrm{du} = \mathrm{dx}$. Therefore, our integral becomes the following.

$$\int_{-1}^{0} x\sqrt{1+x}\ \mathrm{dx} = \int_{0}^{1} (u-1)\sqrt{u}\ \mathrm{du} = \int_{0}^{1} \left(u^{3/2} - u^{1/2}\right)\ \mathrm{du}$$

Integrating with the reverse power rule, we have the following.

$$\int_{0}^{1} \left(u^{3/2} - u^{1/2}\right)\ \mathrm{du} = \left(\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right)\Bigg|_{0}^{1} = \left(\frac{2}{5} - \frac{2}{3}\right) - (0-0) = \frac{6}{15} - \frac{10}{15} = \boxed{-\frac{4}{15}}$$

Problem 17. $\int \frac{8x^3}{1+x^8}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{2\arctan\left(x^4\right) + C}$

We consider re-writing the integral as follows.

$$\int \frac{8x^3}{1+x^8}\ \mathrm{dx} = \int \frac{8x^3}{1+(x^4)^2}\ \mathrm{dx}$$

We can then substitute $u = x^4$, which would allow us to quickly substitute the eighth-power term with something more manageable. This also implies that $\mathrm{du} = 4x^3\ \mathrm{dx}$. Therefore, our integral becomes the following.

$$\int \frac{8x^3}{1+(x^4)^2}\ \mathrm{dx} = \int \frac{24x^3}{1+(x^4)^2}\ \mathrm{dx} = \int \frac{2\ \mathrm{du}}{1+u^2}$$

We immediately recognize this as the anti-derivative of arctan, which allows us to recover the answer by substituting for u as follows.

$$\int \frac{2\ \mathrm{du}}{1+u^2} = 2\arctan(u) + C = \boxed{2\arctan\left(x^4\right) + C}$$

Problem 18. $\int \frac{x^2}{(x-1)(x^2+x+1)}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{1}{3}\ln\left|x^3-1\right| + C}$

We can expand the denominator as follows. Otherwise, one can immediately recognize the familiarity to the difference of cubes factorization.

$$(x-1)(x^2+x+1) = (x^3+x^2+x) - (x^2+x+1) = x^3-1$$

Therefore, our integral becomes the following.

$$\int \frac{x^2}{(x-1)(x^2+x+1)}\ \mathrm{dx} = \int \frac{x^2}{x^3-1}\ \mathrm{dx}$$

We can then utilize a u-substitution of $u = x^3-1$, recognizing that the differential $\mathrm{du} = 3x^2\ \mathrm{dx}$. Therefore, our integral becomes the following.

$$\int \frac{x^2}{x^3-1}\ \mathrm{dx} = \int \frac{\mathrm{du}}{3u} = \frac{1}{3}\ln|u| + C = \boxed{\frac{1}{3}\ln|x^3-1| + C}$$

Problem 19. $\displaystyle\int_0^e x^{(\ln(x))^{-1}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{e^2}$

We first consider an "inverse" u-substitution by considering $x = e^u$, which also implies that $\mathrm{dx} = e^u\ \mathrm{du}$. Therefore, our integral becomes the following.

$$\int_0^e x^{(\ln(x))^{-1}}\ \mathrm{dx} = \int_{-\infty}^1 (e^u)^{(\ln(e^u))^{-1}}\ (e^u\ \mathrm{du}) = \int_{-\infty}^1 (e^u)^{u^{-1}}\ (e^u\ \mathrm{du}) = \int_{-\infty}^1 e^1 e^u\ \mathrm{du} = \int_{-\infty}^1 e^{u+1}\ \mathrm{du}$$

From here, we can simply evaluate the integral as follows.

$$\int_{-\infty}^1 e^{u+1}\ \mathrm{du} = e^{u+1}\Big|_{-\infty}^1 = e^{1+1} - e^{-\infty+1} = e^2 - 0 = \boxed{e^2}$$

Problem 20. $\displaystyle\int_0^{\pi/4} \frac{\sin(2x)}{1+2\sin^2(x)}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\ln(2)}{2}}$

We first obtain the relation between $\cos(2x)$ and $\sin^2(x)$ as follows.

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

Substituting this into our integral, we have the following.

$$\int_0^{\pi/4} \frac{\sin(2x)}{1+2\sin^2(x)}\ \mathrm{dx} = \int_0^{\pi/4} \frac{\sin(2x)}{1+2\left(\frac{1-\cos(2x)}{2}\right)}\ \mathrm{dx} = \int_0^{\pi/4} \frac{\sin(2x)}{2-\cos(2x)}\ \mathrm{dx}$$

We can now consider a u-substitution of the denominator $u = 2 - \cos(2x)$, which also implies that $\mathrm{du} = 2\sin(2x)$. Therefore, our integral becomes the following upon adjusting the bounds of the integral.

$$\int_0^{\pi/4} \frac{\sin(2x)}{2-\cos(2x)}\,\mathrm{dx} = \int_1^2 \frac{\mathrm{du}}{2u} = \frac{\ln|u|}{2}\bigg|_1^2 = \frac{\ln(2)}{2} - \frac{\ln(1)}{2} = \boxed{\frac{\ln(2)}{2}}$$

Problem 21. $\displaystyle\int \ln(1+\tan^2 x)\tan x\,\mathrm{dx}$

Proposed by Katia Avanesov

Solution: $\boxed{\ln^2(\cos x) + C}$

Using laws of logarithms and the trigonometric identity $1 + \tan^2 x = \sec^2 x$, we simplify the integral to obtain

$$\int 2\ln(\sec x)\tan x\,dx$$

Notice that this integral can also be written as:

$$2\int \frac{\ln(\sec x)}{\sec x}\sec x \tan x\,\mathrm{dx}$$

Making use of the reverse chain rule, we see that this expression is the derivative of $\ln^2(\sec x) = \boxed{\ln^2(\cos x)}$. Both forms are valid.

Problem 22. $\displaystyle\int_0^{\infty} \frac{1}{\left(x+\frac{1}{x}\right)^2}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\pi}{4}}$

We consider substituting $x = \tan(\theta)$ to utilize the Pythagorean identities in simplifying. This also implies that $\mathrm{dx} = \sec^2(\theta)\,\mathrm{d}\theta$, which produces the equivalent form of the integral as follows.

$$\int_0^{\infty} \frac{1}{\left(x+\frac{1}{x}\right)^2}\,\mathrm{dx} = \int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{\left(\tan(\theta)+\frac{1}{\tan(\theta)}\right)^2} = \int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{(\tan(\theta)+\cot(\theta))^2} = \int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{\tan^2(\theta)+2+\cot^2(\theta)}$$

From this, we consider splitting the 2 into $1+1$ to induce two separate Pythagorean identities as follows.

$$\int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{\tan^2(\theta)+2+\cot^2(\theta)} = \int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{(\tan^2(\theta)+1)+(\cot^2(\theta)+1)} = \int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{\sec^2(\theta)+\csc^2(\theta)}$$

This can be simplified by converting back to terms of $\sin(\theta)$ and $\cos(\theta)$ as follows.

$$\int_0^{\pi/2} \frac{\sec^2(\theta)\,\mathrm{d}\theta}{\sec^2(\theta)+\csc^2(\theta)}\frac{\cos^2(\theta)}{\cos^2(\theta)} = \int_0^{\pi/2} \frac{1\,\mathrm{d}\theta}{1+\cot^2(\theta)} = \int_0^{\pi/2} \frac{1\,\mathrm{d}\theta}{\csc^2(\theta)} = \int_0^{\pi/2} \sin^2(\theta)\,\mathrm{d}\theta$$

Finally, we can expand using the double-angle formula to obtain the final answer to the integral.

$$\int_0^{\pi/2} \sin^2(\theta)\,\mathrm{d}\theta = \int_0^{\pi/2}\left(\frac{1-\cos(2\theta)}{2}\right)\mathrm{d}\theta = \left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right)\bigg|_0^{\pi/2} = \left(\frac{\pi}{4} - \frac{\sin(\pi)}{4}\right) - (0-0) = \boxed{\frac{\pi}{4}}$$

Problem 23. $\displaystyle\int \frac{\sin^3(x)}{\sqrt{\cos(x)}}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{2}{5}\cos^{5/2}(x) - 2\cos^{1/2}(x) + C}$

We seek to make everything in terms of $\cos(x)$ in order to attempt a u-substitution. Therefore, we split $\sin^3(x)$ as follows to induce a $\cos(x)$.

$$\int \frac{\sin^3(x)}{\sqrt{\cos(x)}}\,\mathrm{dx} = \frac{\sin(x)\sin^2(x)}{\sqrt{\cos(x)}}\,\mathrm{dx} = \frac{\sin(x)(1-\cos^2(x))}{\sqrt{\cos(x)}}\,\mathrm{dx}$$

To remove the issue raised by the denominator, we consider $u = \cos(x)$. This implies that $\mathrm{du} = -\sin(x)$, and produces the equivalent form of the integral as follows.

$$\int \frac{\sin(x)(1-\cos^2(x))}{\sqrt{\cos(x)}}\,\mathrm{dx} = \int \frac{(-\mathrm{du})(1-u^2)}{\sqrt{u}} = \int \frac{(u^2-1)}{\sqrt{u}}\,\mathrm{du}$$

Splitting the integrand, we get the following simplification upon integrating.

$$\int \frac{(u^2-1)}{\sqrt{u}}\,\mathrm{du} = \int \left(\frac{u^2}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)\,\mathrm{du} = \int \left(u^{3/2} - u^{-1/2}\right)\,\mathrm{du} = \frac{u^{3/2+1}}{\frac{3}{2}+1} - \frac{u^{-1/2+1}}{\frac{-1}{2}+1} + C = \frac{2}{5}u^{5/2} - 2u^{1/2} + C$$

Finally, we reverse the u-substitution to recover the solution in terms of x as follows.

$$\frac{2}{5}u^{5/2} - 2u^{1/2} + C = \boxed{\frac{2}{5}\cos^{5/2}(x) - 2\cos^{1/2}(x) + C}$$

Problem 24. $\displaystyle\int_1^{\sqrt[10]{2}} \frac{1}{x(x^{10}+1)}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{1}{10}\ln\left(\frac{4}{3}\right)}$

The bounds give us a clear indication that we should consider the u-substitution of $u = x^{10}$. This implies that $\mathrm{du} = 10x^9$ dx. Substituting this into our integral, we have the following.

$$\int_1^{\sqrt[10]{2}} \frac{1}{x(x^{10}+1)}\,\mathrm{dx} = \int_1^2 \frac{1}{\sqrt[10]{u}(u+1)}\left(\frac{\mathrm{du}}{10\sqrt[10]{u^9}}\right) = \frac{1}{10}\int_1^2 \frac{\mathrm{du}}{u(u+1)}$$

Performing partial fraction decomposition on the integrand, we immediately recognize that $A = 1$ and $B = -1$. Therefore, we have the following.

$$\frac{1}{10}\int_1^2 \frac{\mathrm{du}}{u(u+1)} = \frac{1}{10}\int_1^2 \left(\frac{A}{u} + \frac{B}{u+1}\right)\,\mathrm{du} = \frac{1}{10}\int_1^2 \left(\frac{1}{u} - \frac{1}{u+1}\right)\,\mathrm{du} = \frac{1}{10}(\ln|u| - \ln|u+1|)\Big|_1^2$$

Simplifying, we have the following.

$$\frac{1}{10}\ln\left(\frac{u}{u+1}\right)\Big|_1^2 = \frac{1}{10}\left(\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{2}\right)\right) = \frac{1}{10}\ln\left(\frac{2}{3}\frac{2}{1}\right) = \boxed{\frac{1}{10}\ln\left(\frac{4}{3}\right)}$$

Problem 25. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos(x)(1+\arctan(x))}{2-\cos^2(x)}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{\pi}{2}}$

We begin by splitting the integral as follows in the numerator. Since $\arctan(x)$ is an odd function, then the integral will necessarily evaluate to 0 over symmetric bounds.

$$\int_{-\pi/2}^{\pi/2} \frac{\cos(x)(1+\arctan(x))}{2-\cos^2(x)} = \int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{2-\cos^2(x)}\ \mathrm{dx} + \int_{-\pi/2}^{\pi/2} \frac{\cos(x)\arctan(x)}{2-\cos^2(x)}\ \mathrm{dx} = \int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{2-\cos^2(x)}\ \mathrm{dx}$$

Using the Pythagorean identity on $\cos^2(x)$, we have the following.

$$\int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{2-\cos^2(x)}\ \mathrm{dx} = \int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{2-(1-\sin^2(x))} = \int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{1+\sin^2(x)}\ \mathrm{dx}$$

We can then utilize a u-substitution of $u = \sin(x)$, which gives us the following.

$$\int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{1+\sin^2(x)}\ \mathrm{dx} = \int_{-1}^{1} \frac{1}{1+u^2}\ \mathrm{du} = \arctan(u)\Big|_{-1}^{1} = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \boxed{\frac{\pi}{2}}$$

Problem 26. $\displaystyle\int \frac{\cos(x)-\sin(2x)}{\sin(x)}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\ln|\sin(x)| - 2\sin(x) + C}$

Expanding $\sin(2x)$ by the double-angle formula and splitting the integrand, we have the following equivalence.

$$\int \frac{\cos(x)-\sin(2x)}{\sin(x)}\ \mathrm{dx} = \int \frac{\cos(x)-2\sin(x)\cos(x)}{\sin(x)}\ \mathrm{dx} = \int \left(\cot(x) - 2\cos(x)\right)\ \mathrm{dx}$$

We can now integrate regularly on each of the components of the integrand as follows.

$$\int \left(\cot(x) - 2\cos(x)\right)\ \mathrm{dx} = \int \cot(x)\ \mathrm{dx} - 2\int \cos(x)\ \mathrm{dx} = \boxed{\ln|\sin(x)| - 2\sin(x) + C}$$

Problem 27. $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{-2\cos(\sqrt{x}) + C}$

To remove the inner argument of $\sqrt{x}$, we consider a u-substitution of $u = \sqrt{x}$, which implies that $\mathrm{du} = 1/(2\sqrt{x})\ \mathrm{dx}$. Therefore, our integral becomes the following.

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\ \mathrm{dx} = \int \frac{\sin(u)}{u}\ (2u\ \mathrm{du}) = \int 2\sin(u)\ \mathrm{du}$$

Integrating, we can then reverse our u-substitution to recover the solution in terms of x as follows.

$$\int 2\sin(u)\ \mathrm{du} = -2\cos(u) + C = \boxed{-2\cos(\sqrt{x}) + C}$$

Problem 28. $\displaystyle\int_0^\infty \frac{x^2-1}{x^4+3x^2+1}\mathrm{dx}$

Proposed by Katia Avanesov

Solution: $\boxed{\arctan\left(x+\dfrac{1}{x}\right)+C}$

Notice that the denominator doesn't factor nicely if we take it all as one expression, so instead, we can write

$$\int \frac{x^2-1}{x^2+(x^2+1)^2}\mathrm{dx}$$

Dividing through by x^2,

$$\int \frac{1-\frac{1}{x^2}}{1+(x+\frac{1}{x})^2}\mathrm{dx}$$

Which is now in a familiar format for the substitution $u = x+\frac{1}{x}$ such that our integral transforms to

$$\int \frac{1}{1+u^2}\mathrm{du} = \arctan u + C = \boxed{\arctan\left(x+\frac{1}{x}\right)+C}$$

(Alternative form: since $\arctan(x)+\arctan(\frac{1}{x}) = \frac{\pi}{2}$, the expression $C-\arctan(\frac{x}{1+x^2})$ is also valid).

Problem 29. $\displaystyle\int \frac{x-1}{x^2-x\ln(x)}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\ln|x-\ln(x)|+C}$

We begin by factoring the denominator of the integrand as follows.

$$\int \frac{x-1}{x^2-x\ln(x)}\ \mathrm{dx} = \int \frac{x-1}{x(x-\ln(x))}\ \mathrm{dx}$$

To remove the $\ln(x)$ present in the denominator, we consider the u-substitution of $u = x-\ln(x)$, which implies the following differential.

$$\mathrm{du} = \left(1-\frac{1}{x}\right)\ \mathrm{dx} = \frac{x-1}{x}\ \mathrm{dx}$$

Therefore, the equivalent form of our integral becomes the following.

$$\int \frac{x-1}{x(x-\ln(x))}\ \mathrm{dx} = \int \frac{1}{(x-\ln(x))}\frac{x-1}{x}\ \mathrm{dx} = \int \frac{\mathrm{du}}{u} = \ln|u|+C = \boxed{\ln|x-\ln(x)|+C}$$

Problem 30. $\int_0^1 \frac{x\ln x}{1-x^2}\ \mathrm{dx}$

Proposed By Katia Avanesov

Solution: $\boxed{-\frac{\pi^2}{24}}$

Let us proceed by integration by parts where we let $f(x)=\ln x$ which implies $f'(x)=\frac{1}{x}$ and $g'(x)=\frac{x}{1-x^2}$, which implies $g(x)=-\frac{1}{2}\ln(1-x^2)$. So,

$$\int_0^1 \frac{x\ln x}{1-x^2}\mathrm{dx} = -\frac{1}{2}\ln(x)\ln(1-x^2)\Big]_0^1 + \frac{1}{2}\int_0^1 \frac{1}{x}\ln(1-x^2)\mathrm{dx}$$

The first term on the RHS evaluates to 0. We can write $\ln(1-x^2)$ as a Mclaurin series to obtain:

$$\int_0^1 \frac{x\ln x}{1-x^2}\mathrm{dx} = -\frac{1}{2}\int_0^1 \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^{2k}}{k}\mathrm{dx}$$

Reversing the order of summation and integration, we can integrate simply using the power rule to obtain

$$\int_0^1 \frac{x\ln x}{1-x^2}\mathrm{dx} = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k}}{2k^2}\Big]_0^1$$

It is known that $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$, so the result then evaluates to $\boxed{-\frac{\pi^2}{24}}$.

Medium

Problem 31. $\int x^{x^2+1}\left(2\ln(x)+1\right)\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{x^{x^2}+C}$

To remove the x^2 in the exponent, we consider the reverse u-substitution of $x=\sqrt{u}$, which then implies that $\mathrm{dx}=1/(2\sqrt{u})$. Therefore, we have the following equivalent form of the integral.

$$\int x^{x^2+1}\left(2\ln(x)+1\right)\ \mathrm{dx} = \int \left(\sqrt{u}\right)^{(\sqrt{u})^2+1}\left(2\ln(\sqrt{u})+1\right)\ \left(\frac{1}{2\sqrt{u}}\ \mathrm{du}\right)$$

Simplifying, we have the following.

$$\int \left(\sqrt{u}\right)^{u+1}\left(\ln(u)+1\right)\ \left(\frac{1}{2\sqrt{u}}\ \mathrm{du}\right) = \int \frac{u^{u/2}}{2}\left(\ln(u)+1\right)\ \mathrm{du}$$

We can now consider the substitution of $v=u^{u/2}$, as the integrand potentially resembles the derivative of this. Computing dv, we have the following.

$$\ln(v)=\frac{u}{2}\ln(u)$$

$$\frac{\mathrm{dv}}{v} = \left(\frac{\ln(u)}{2} + \frac{1}{2}\right)\ \mathrm{du}$$

$$\mathrm{dv} = \left(u^{u/2}\left(\frac{\ln(u)}{2} + \frac{1}{2}\right)\right)\ \mathrm{du} = \frac{u^{u/2}}{2}\left(\ln(u)+1\right)\ \mathrm{du}$$

Therefore, our integral becomes the following, in which we invert our previous substitutions to recover the solution in terms of x.

$$\int \frac{u^{u/2}}{2}\left(\ln(u)+1\right)\ \mathrm{du} = \int\ \mathrm{dv} = v + C = u^{u/2} + C = (x^2)^{x^2/2} + C = x^{2x^2/2} + C = \boxed{x^{x^2} + C}$$

Problem 32. $\displaystyle\int_0^1 \frac{e^{\arctan(x)}}{(x^2+1)^{3/2}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\dfrac{e^{\pi/4}}{\sqrt{2}} - \dfrac{1}{2}}$

Motivated by both the $\arctan(x)$ present and the x^2+1 term, we consider the substitution of $x = \tan(\theta)$, which implies that $\mathrm{dx} = \sec^2(\theta)\ \mathrm{d}\theta$. Therefore, we get the equivalent form of the integral as follows.

$$\int_0^1 \frac{e^{\arctan(x)}}{(x^2+1)^{3/2}}\ \mathrm{dx} = \int_0^{\pi/4} \frac{e^{\arctan(\tan(\theta))}}{(\tan^2(\theta)+1)^{3/2}}\ (\sec^2(\theta)\ \mathrm{d}\theta)$$

Simplifying, we have the following. We additionally note that $\arctan(\tan(\theta)) = \theta$ as the $\theta_1 = \arctan(0) = 0$ and $\theta_2 = \arctan(1) = \pi/4$ lie within the range of $[-\pi/2, \pi/2]$ for which the identity holds.

$$\int_0^{\pi/4} \frac{e^{\arctan(\tan(\theta))}}{(\tan^2(\theta)+1)^{3/2}}\ (\sec^2(\theta)\ \mathrm{d}\theta) = \int_0^{\pi/4} \frac{e^{\theta}}{(\sec^2(\theta))^{3/2}}\ (\sec^2(\theta)\ \mathrm{d}\theta) = \int_0^{\pi/4} e^{\theta}\cos(\theta)\ \mathrm{d}\theta$$

We can use integration by parts on this, with $a = \cos(\theta)$ and $\mathrm{db} = e^{\theta}\ \mathrm{d}\theta$, which gives the following.

$$\int_0^{\pi/4} e^{\theta}\cos(\theta)\ \mathrm{d}\theta = \int a\ \mathrm{db} = ab - \int b\ \mathrm{da} = (\cos(\theta)e^{\theta})\Big|_0^{\pi/4} + \int_0^{\pi/4} e^{\theta}\sin(\theta)\ \mathrm{d}\theta$$

Doing integration by parts on this integral again, with $b = \sin(\theta)$ and $\mathrm{da} = e^{\theta}\ \mathrm{d}\theta$, we have the following.

$$\int_0^{\pi/4} \sin(\theta)e^{\theta}\ \mathrm{d}\theta = \int b\ \mathrm{da} = ab - \int a\ \mathrm{db} = (\sin(\theta)e^{\theta})\Big|_0^{\pi/4} - \int_0^{\pi/4} \cos(\theta)e^{\theta}\ \mathrm{d}\theta$$

Therefore, we have the following equivalence.

$$\int_0^{\pi/4} e^{\theta}\cos(\theta)\ \mathrm{d}\theta = (\cos(\theta)e^{\theta})\Big|_0^{\pi/4} + (\sin(\theta)e^{\theta})\Big|_0^{\pi/4} - \int_0^{\pi/4} \cos(\theta)e^{\theta}\ \mathrm{d}\theta$$

Simplifying, we obtain the final answer for the desired integral.

$$2\int_0^{\pi/4} e^{\theta}\cos(\theta)\ \mathrm{d}\theta = \left(\frac{\sqrt{2}}{2}e^{\pi/4} - 1\right) + \left(\frac{\sqrt{2}}{2}e^{\pi/4} - 0\right) = \sqrt{2}e^{\pi/4} - 1$$

$$\int_0^{\pi/4} e^{\theta}\cos(\theta)\ \mathrm{d}\theta = \boxed{\frac{e^{\pi/4}}{\sqrt{2}} - \frac{1}{2}}$$

Problem 33. $\int_0^{2\pi} \sin(\sin(x) - x)\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{0}$

We can utilize King's Rule, which is written below.

$$\int_a^b f(x)\ \mathrm{dx} = \int_a^b f(a+b-x)\ \mathrm{dx}$$

Thus, we label the initial integral as I and use King's Rule to get an alternate integral as follows.

$$I = \int_0^{2\pi} \sin(\sin(x) - x)\ \mathrm{dx} = \int_0^{2\pi} \sin(\sin(2\pi - x) - (2\pi - x))\ \mathrm{dx}$$

Simplifying the inner argument, we have the following.

$$\sin(2\pi - x) - (2\pi - x) = \sin(2\pi)\cos(x) - \sin(x)\cos(2\pi) - 2\pi + x = -\sin(x) - 2\pi + x$$

Adding the two forms of the integral together, we have the following.

$$2I = \int_0^{2\pi} \left(\sin(\sin(x) - x) + \sin(-\sin(x) - 2\pi + x)\right)\ \mathrm{dx}$$

We can now use the identity that $\sin(x \pm 2\pi k) = \sin(x)$ for $k \in \mathbb{Z}$ to simplify once again.

$$2I = \int_0^{2\pi} \left(\sin(\sin(x) - x) + \sin(-\sin(x) + x)\right)\ \mathrm{dx}$$

Finally, we can recognize that the former part of the integrand is exactly opposite to that of the latter part of the integrand due to $\sin(x)$ being odd, as shown below.

$$\sin(-\sin(x) + x) = \sin(-(\sin(x) - x)) = -\sin(\sin(x) - x)$$

Therefore, our integral simplifies and allows us to deduce the value of I, as shown below.

$$2I = \int_0^{2\pi} \left(\sin(\sin(x) - x) - \sin(\sin(x) - x)\right)\ \mathrm{dx} = \int_0^{2\pi} 0\ \mathrm{dx} = 0$$

$$I = \boxed{0}$$

Problem 34. $\int_{-\infty}^{\infty} e^{-x^2} \arctan(e^{2x})\ \mathrm{dx}$

Proposed by Katia Avanesov

Solution: $\boxed{\dfrac{\pi\sqrt{\pi}}{4}}$

Let I be the initial integral. Using the additive property of integrals, split up I into two integrals:

$$I = \int_{-\infty}^{0} e^{-x^2} \arctan(e^{2x})\ \mathrm{dx} + \int_0^{\infty} e^{-x^2} \arctan(e^{2x})\ \mathrm{dx}$$

Let us apply the substitution $u=-x$ to the first integral to obtain

$$I=\int_0^{\infty} e^{-x^2}\arctan(e^{-2x})\text{ dx}+\int_0^{\infty} e^{-x^2}\arctan(e^{2x})\text{ dx}$$

Now, we can recombine the integrals as they have the same bounds. We find:

$$I=\int_0^{\infty} e^{-x^2}(\arctan(e^{-2x})+\arctan(e^{2x}))\text{ dx}$$

Applying the identity $\arctan(x)+\arctan(\frac{1}{x})=\frac{\pi}{2}$, we simplify the integral to

$$I=\frac{\pi}{2}\int_0^{\infty} e^{-x^2}\text{ dx}$$

The integral is then just the well known Gaussian integral which yields the standard result of $\frac{\sqrt{\pi}}{2}$. Hence, by multiplying this result by the factor of $\frac{\pi}{2}$, we find that $\boxed{I=\frac{\pi\sqrt{\pi}}{4}}$

Problem 35. $\displaystyle\int_0^{\infty}\frac{\{x\}^{\lceil x\rceil}}{1+\lceil x\rceil}\text{ dx}$

Proposed by Katia Avanesov

Solution: $\boxed{\frac{\pi^2}{6}-1}$

Label our integral I. Recall that $\{x\}=x-\lfloor x\rfloor$. Also, by the additive property of integrals, we can rewrite I as

$$\sum_{k=0}^{\infty}\int_k^{k+1}\frac{(x-k)^{k+1}}{k+2}\text{ dx}$$

Which we can easily integrate to give

$$\sum_{k=0}^{\infty}\frac{(x-k)^{k+2}}{(k+2)^2}\Big]_k^{k+1}$$

Evaluating the bounds, we find,

$$\sum_{k=0}^{\infty}\frac{1}{(k+2)^2}$$

But this is just $(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots)-1$. The first term is known to be $\frac{\pi^2}{6}$, which immediately gives that $\boxed{I=\frac{\pi^2}{6}-1}$

Problem 36. $\displaystyle\int_0^{\infty}\frac{x}{e^x-1}\text{dx}$

Proposed by Sam Murray

Solution: $\boxed{\frac{\pi^2}{6}}$

The first of our "tricks" herein will be multiplying by 1, in a special form:

$$\int_0^\infty \frac{x}{e^x-1}\,\mathrm{dx} = \int_0^\infty \frac{x}{e^x-1}\frac{e^{-x}}{e^{-x}}\,\mathrm{dx} = \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,\mathrm{dx}.$$

Now, adding 0 in the numerator in, again, a special form:

$$\int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,\mathrm{dx} = \int_0^\infty x\left(\frac{e^{-x}-1+1}{1-e^{-x}}\right)\mathrm{dx} = \int_0^\infty x\left(-1+\frac{1}{1-e^{-x}}\right)\mathrm{dx}.$$

Recalling the geometric series relation

$$\sum_{k=0}^{\infty} e^{-kx} = \frac{1}{1-e^{-x}}$$

with the upheld criterion that $|e^{-x}|<1$ for relevant x allows the following.

$$\int_0^\infty x\left(-1+\frac{1}{1-e^{-x}}\right)\mathrm{dx} = \int_0^\infty x\left(-1+\sum_{k=0}^{\infty} e^{-kx}\right)\mathrm{dx} = \int_0^\infty \sum_{k=1}^{\infty} xe^{-kx}\,\mathrm{dx}.$$

The dominated convergence theorem allows the interchange of the discrete and continuous summation operators so that we gain a new integration problem easily done via integration by parts. Accordingly,

$$\sum_{k=1}^{\infty}\int_0^\infty xe^{-kx}\,\mathrm{dx} = \lim_{b\to\infty}\sum_{k=1}^{\infty}\left(-\frac{xe^{-kx}}{k}\bigg|_0^b + \int_0^\infty \frac{e^{-kx}}{k}\,\mathrm{dx}\right) = \lim_{b\to\infty}\sum_{k=1}^{\infty}\left(0-\frac{e^{-kx}}{k^2}\bigg|_0^b\right) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \boxed{\frac{\pi^2}{6}},$$

as known from the Basel problem.

Problem 37. $\int_0^{\frac{\pi}{2}} \min\{\sin(x),\cot(x)\}\,\mathrm{dx}$

Proposed by Sam Murray

Solution: $\boxed{\frac{3-\sqrt{5}}{2}-\frac{1}{2}\ln\left(\frac{-1+\sqrt{5}}{2}\right)}$

Obviously, by the nature of the integrand, we want to conclude on what intervals one continuous function is less than the other. We may do this by finding the location at which they intersect and subsequently testing points in a neighborhood about it. Foremost,

$$\sin(x)=\cot(x) \implies \sin^2(x)=\cos(x) \implies \cos^2(x)+\cos(x)-1=0 \implies \cos(x)=\frac{-1\pm\sqrt{5}}{2}.$$

We discount the negative conjugate in preparation for operating with $\arccos(\cdot)$, yielding

$$x=\arccos\left(\frac{-1+\sqrt{5}}{2}\right),$$

which is the only intersection on $[0, \frac{\pi}{2}]$ since cosine is 2π-periodic. We further intuit that $\sin(x) < \cot(x)$ occurs for all $x > 0$ preceding the intersection since $\sin(0) = 0$. The other x succeeding the intersection has that $\sin(x) > \cot(x)$ by testing, say, $x = \frac{\pi}{4}$. With this information, we may deduce

$$\int_0^{\frac{\pi}{2}} \min\{\sin(x), \cot(x)\} \text{ dx} = \int_0^{\arccos\left(\frac{-1+\sqrt{5}}{2}\right)} \sin(x) \text{ dx} + \int_{\arccos\left(\frac{-1+\sqrt{5}}{2}\right)}^{\frac{\pi}{2}} \cot(x) \text{ dx}.$$

We begin with the first integral on the RHS:

$$\int_0^{\arccos\left(\frac{-1+\sqrt{5}}{2}\right)} \sin(x) \text{ dx} = \cos(0) - \cos\left(\arccos\left(\frac{-1+\sqrt{5}}{2}\right)\right) = \frac{3-\sqrt{5}}{2}.$$

Now the second:

$$\int_{\arccos\left(\frac{-1+\sqrt{5}}{2}\right)}^{\frac{\pi}{2}} \cot(x) \text{ dx} = \ln\left|\sin\left(\frac{\pi}{2}\right)\right| - \ln\left|\sin\left(\arccos\left(\frac{-1+\sqrt{5}}{2}\right)\right)\right| = -\frac{1}{2}\ln\left(\frac{-1+\sqrt{5}}{2}\right).$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \min\{\sin(x), \cot(x)\} \text{ dx} = \boxed{\frac{3-\sqrt{5}}{2} - \frac{1}{2}\ln\left(\frac{-1+\sqrt{5}}{2}\right)}.$$