Individual Round 2023-2024

Problem 1. A standard analog clock has three hands (seconds, minute, and hour), and the usual 12 hourly labels 1:00, 2:00, ..., 12:00. Each hand rotates around the clock continuously. Assume that this clock records the time perfectly. At 12:00 PM sharp, the three hands of the clock are all pointing in exactly the same direction, at the 12:00 label. After some exact amount of time passes during the same day, the minute and hour hands are again pointing in the exact same direction, while the seconds hand is pointing in a direction closer to the 4:00 label than any of the other hourly labels. How much time, in minutes, has passed? Round your answer to the nearest integer.

Problem 2. The dreaded pirate captain Jack D. Luffy and his crew found a buried treasure chest on a deserted island, containing gold and silver coins as loot. Monetarily, a single gold coin is worth $Q > 1$ silver coins (Q need not be an integer), and a single silver coin is worth many times one unit of the national currency. The pirates agree to share the spoils fairly; that is, every pirate receives the same amount of loot in monetary value (but the number of gold and silver coins received by each pirate may vary). Jack receives $\frac{1}{6}$ the total number of gold coins and $\frac{1}{10}$ the total number of silver coins from the chest. After counting up his loot again, Jack observes he has received four times as many silver coins as gold coins. What is the sum of all possible values of Q?

Problem 3. A Caltech prefrosh is participating in rotation. There are 8 houses at Caltech: Avery, Blacker, Dabney, Fleming, Lloyd, Page, Ricketts, and Venerable. The prefrosh visits these houses in some order, each of them exactly once. Throughout rotation, the prefrosh maintains a ranking list of all of the houses that the prefosh has visited. After every visit to a house, the prefrosh updates the ranking list by inserting the most recently visited house to either the top or the bottom of the list, each with probability $\frac{1}{2}$, while keeping the order of all previously visited houses the same. Compute the probability that at the end of rotation, third house the prefrosh visited is not ranked fourth or fifth on their list.

Problem 4. Find the number of positive real numbers x such that

$$\log_3(x) = \frac{x+15}{4} - \left\lfloor \frac{x-1}{4} \right\rfloor \qquad \textbf{and} \qquad \lfloor \log_5(x) \rfloor = 2$$

(for any positive real number y, recall that $\lfloor y \rfloor$ is the greatest integer less than or equal to y).

Problem 5. Call a natural number $n > 1$ *flavorful* if, for every prime divisor p of n, p^2 is also a divisor of n. Find the largest positive integer that cannot be expressed as the sum of one or more distinct flavorful numbers.

Problem 6. Let $\mathscr{R}$ be a right rectangular prism with vertices $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$, where $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two parallel rectangular faces, with $A_1A_2 = B_1B_2 = 3, A_2A_3 = B_2B_3 = 7$, and $\overline{A_1B_1}, \overline{A_2B_2}, \overline{A_3B_3}$, and $\overline{A_4B_4}$ are mutually parallel edges of $\mathscr{R}$. Suppose that a plane intersects segments $\overline{A_1B_1}, \overline{A_2B_2}, \overline{A_3B_3}$, and $\overline{A_4B_4}$ at P_1, P_2, P_3, and P_4, respectively, dividing $\mathscr{R}$ into two solids, each with volume exactly $\frac{1}{2}$ that of $\mathscr{R}$. If three of the lengths A_1P_1, A_2P_2, A_3P_3, and A_4P_4 are $3, 4$, and 6 in some order, then find the sum of all possible values of the volume of $\mathscr{R}$.

Problem 7. League of Legends is a two-team video game, one team playing on the *blue side* and the other playing on the *red side*, where every game results in a win for one team and a loss for the other. The League of Legends teams T1 and JDG play a best-of-five series of games: that is, the two teams play games until one of them has won three games. In the first game, T1 plays on the blue side. In every subsequent game, the team that lost the previous game plays on the blue side. The two teams are equally matched, but "side selection" matters: the probability that the team on the blue side wins any particular game is $\frac{2}{3}$. After the best-of-five series, what is the expected number of games won by the team playing on the blue side?

Problem 8. For a positive integer $k \geq 2$, let α_k, β_k, γ_k be the complex roots (with multiplicity) of the cubic equation $\left(x - \frac{1}{k-1}\right)\left(x - \frac{1}{k}\right)\left(x - \frac{1}{k+1}\right) = \frac{1}{k}$. Determine the value of

$$\sum_{k=2}^{\infty} \frac{\alpha_k \beta_k \gamma_k \cdot (1+\alpha_k) \cdot (1+\beta_k) \cdot (1+\gamma_k)}{k+1}.$$

Problem 9. Let ABC be a triangle with orthocenter H and $AB = 17, BC = 28, CA = 25$. Let X be a point whose distance to $\overline{BC}$ is 2. Suppose $\overline{BX}$ and $\overline{HC}$ intersects at Y, and $\overline{CX}$ and $\overline{HB}$ intersects at Z, such that $YZ < BC$ and $YC \cdot CA = ZB \cdot BA$. Find AX.

Problem 10. Brian and Stephanie are sitting next to each other at a round table with a number of other people (possibly 0 other people). The people at the table pass a rubber ball to each other, always to the person on their left. The ball starts with Stephanie, and arrives at Brian after exactly 2024 passes.

Suppose that after k more passes ($1 \leq k \leq 2024$), Stephanie receives the ball again. How many possible values of k are there?

Problem 11. Pick a point P uniformly at random from the interior of an equilateral triangle ABC. What is the probability that the lengths PA, PB, PC determine a non-degenerate triangle of area at least $\frac{2}{9}$ that of triangle ABC?

Problem 12. Compute

$$\sum_{k=0}^{103} \left(\lfloor k(3-\sqrt{3}) \rfloor - \left\lfloor \left\lfloor (2-\sqrt{3})(k+1) \right\rfloor \cdot (3+\sqrt{3}) \right\rfloor \right)^2$$

(note that $\sqrt{3} \approx 1.73$).

Problem 13. Let N be the number of distinct tuples $(x_1, x_2, \ldots, x_{46})$ of positive integers with $x_1, x_2, \ldots, x_{46} \leq 88$ such that the remainder when $x_1^{35} + x_2^{35} + \cdots + x_{46}^{35}$ is divided by 2024 is 253. Compute the remainder when N is divided by 46.

Problem 14. Let ABC be an acute triangle with $AC > AB$. Let D, E, and F be the feet of the altitudes from A, B, and C, onto $\overline{BC}, \overline{CA}$ and $\overline{AB}$, respectively. Let K be the intersection of $\overline{EF}$ and $\overline{AD}$, and let I be the intersection of $\overline{EF}$ and $\overline{BC}$. Let W be a point on ray $\overrightarrow{BF}$ such that $\angle IWF = \angle FWE$. Suppose $AKBI$ is cyclic and $\cos(\angle BCA) = \frac{2}{5}$. Find $\frac{WB}{WI}$.

Problem 15. A function $f : \{1,2,3,4,5,6,7\} \to \{2,3,4,5,6\}$ is called *roughly monotonic* if for every integer $2 \leq k \leq 5$ in which there is some integer $1 \leq a \leq 7$ with $f(a) = k$, then there exist integers $1 \leq a_0 < b \leq 7$ such that $f(a_0) = k, f(b) = k+1$. Compute the number of roughly monotonic functions f.