CMM 2024 Integration Bee Qualification Test Solutions

Problem 1. $\int_0^1 (4x - 6x^{2/3})\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{-8}{5}}$

This is a simple use of the power rule for integrals.

$$\int_0^1 (4x-6x^{2/3})\ \mathrm{dx} = \left(\frac{4\cdot x^{1+1}}{1+1} - \frac{6\cdot x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right)\Bigg|_0^1 = \left(2x^2 - \frac{18}{5}\cdot x^{5/3}\right)\Bigg|_0^1 = 2\cdot(1)^2 - \frac{18}{5}\cdot 1^{5/3} = 2 - \frac{18}{5} = \boxed{\frac{-8}{5}}$$

Problem 2. $\int_1^4 \frac{x\sqrt{x}-1}{x}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{14}{3} - 2\ln(2)}$

We begin by splitting the integrand into two fractions as follows.

$$\int_1^4 \frac{x\sqrt{x}-1}{x}\ \mathrm{dx} = \int_1^4 \left(\sqrt{x} - \frac{1}{x}\right)\ \mathrm{dx}$$

We can now integrate separately as follows.

$$\int_1^4 \left(\sqrt{x} - \frac{1}{x}\right)\ \mathrm{dx} = \int_1^4 \sqrt{x}\ \mathrm{dx} - \int_1^4 \frac{1}{x}\ \mathrm{dx} = \left(\frac{2x^{3/2}}{3}\right)\Bigg|_1^4 - (\ln|x|)\Bigg|_1^4$$

We now evaluate each of these expressions as follows.

$$\left(\frac{2x^{3/2}}{3}\right)\Bigg|_1^4 - (\ln|x|)\Bigg|_1^4 = \left(\frac{2\cdot 4^{3/2}}{3} - \frac{2\cdot 1^{3/2}}{3}\right) - (\ln(4) - \ln(1)) = \left(\frac{16}{3} - \frac{2}{3}\right) - (2\ln(2) - 0) = \boxed{\frac{14}{3} - 2\ln(2)}$$

Problem 3. $\int_0^{\pi/4} \frac{2\tan(\theta)}{1+\tan^2(\theta)}\ \mathrm{d}\theta$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{1}{2}}$

We first employ the variant of the Pythagorean identity that states the following.

$$1+\tan^2(\theta) = \sec^2(\theta)$$

Therefore, the integrand simplifies as follows.

$$\int_0^{\pi/4} \frac{2\tan(\theta)}{1+\tan^2(\theta)}\ \mathrm{d}\theta = \int_0^{\pi/4} \frac{2\tan(\theta)}{\sec^2(\theta)}\ \mathrm{d}\theta = \int_0^{\pi/4} 2\tan(\theta)\cos^2(\theta)\ \mathrm{d}\theta$$

Expanding the definition of tangent and double-angle identity, we have the following.

$$\int_0^{\pi/4} 2 \cdot \left(\frac{\sin(\theta)}{\cos(\theta)}\right) \cdot \cos^2(\theta)\ \mathrm{d}\theta = \int_0^{\pi/4} 2\sin(\theta)\cos(\theta)\ \mathrm{d}\theta = \int_0^{\pi/4} \sin(2\theta)\ \mathrm{d}\theta$$

We can now integrate regularly to obtain a final answer as follows.

$$\int_0^{\pi/4} \sin(2\theta)\ \mathrm{d}\theta = \frac{-\cos(2\theta)}{2} = \Big|_0^{\pi/4} = \frac{-\cos(\pi/2)}{2} + \frac{\cos(0)}{2} = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

Problem 4. $\int_1^{\infty} \frac{e^{1/x}}{x^2}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{e-1}$

To get rid of the exponent, we immediately consider the u-substitution of $u = 1/x$, which implies that $\mathrm{du} = -1/x^2\ \mathrm{dx}$. Therefore, our integral becomes the following.

$$\int_1^{\infty} \frac{e^{1/x}}{x^2} = \int_1^0 -e^u\ \mathrm{du} = \int_0^1 e^u\ \mathrm{du}$$

Note that our bounds were transformed as follows.

$$u = \frac{1}{1} = 1 \qquad\qquad u = \frac{1}{\infty} = 0$$

We can then evaluate this integral to get our final answer.

$$\int_0^1 e^u\ \mathrm{du} = e^u \Big|_0^1 = e^1 - e^0 = \boxed{e-1}$$

Problem 5. $\int_0^{2^{-1/4}} \frac{2x}{\sqrt{1-x^4}}\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{\pi}{4}}$

We first recognize that $2x$ is the derivative of x^2, which means we can utilize the u-substitution of $u = x^2$ as follows.

$$\int_0^{2^{-1/4}} \frac{2x}{\sqrt{1-x^4}}\ \mathrm{dx} = \int_0^{2^{-1/2}} \frac{\mathrm{du}}{\sqrt{1-u^2}}$$

Note that our bounds were transformed as follows.

$$u = 0^2 = 0$$

$$u = \left(2^{-1/4}\right)^2 = 2^{-1/2}$$

One can immediately recognize that this is the anti-derivative of $\arcsin(u)$, but we can show it here by considering the substitution $u = \sin(v)$, which makes $\mathrm{du} = \cos(v)\ \mathrm{dv}$ as follows.

$$\int_0^{2^{-1/2}} \frac{\mathrm{du}}{\sqrt{1-u^2}} = \int_{\arcsin(0)}^{\arcsin\left(2^{-1/2}\right)} \frac{\cos(v)\ \mathrm{dv}}{\sqrt{1-\sin^2(v)}} = \int_0^{\arcsin\left(2^{-1/2}\right)} \frac{\cos(v)\ \mathrm{dv}}{\sqrt{\cos^2(v)}} = \int_0^{\arcsin\left(2^{-1/2}\right)} \frac{\cos(v)\ \mathrm{dv}}{|\cos(v)|}$$

To determine if we need to consider sign in the denominator of the integrand, we must evaluate the upper bound of the integral as follows.

$$\arcsin\left(2^{-1/2}\right) = \arcsin\left(\frac{1}{\sqrt{2}}\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Since $\cos(v)$ is positive on the interval elicited by the bounds of the integral, we obtain the final answer as follows.

$$\int_0^{\pi/4} \frac{\cos(v)\ \mathrm{dv}}{\cos(v)} = \int_0^{\pi/4} \mathrm{dv} = v\Big|_0^{\pi/4} = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

Problem 6. $\int_0^6 |x-3|\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{9}$

This integral lends itself to a more geometric approach, in that the integrand can be decomposed into two triangles. Specifically, we have that one of the triangles is sloping downward with $m = -1$ on the interval to $[0,3]$, and one of the triangles is sloping upward with $m = 1$ on the interval to $[3,6]$.

Therefore, we have two triangles, one on each sub-interval. Each of the triangles has a base of 3 since the sub-interval lengths are each 3, and height 3 since the y-intercept of the lines mentioned above is 3. Thus, adding the areas of both of these triangles, we have the following.

$$\frac{9}{2} + \frac{9}{2} = \boxed{9}$$

The calculus-based approach is shown as follows.

$$\int_0^6 |x-3|\ \mathrm{dx} = \int_0^3 -(x-3)\ \mathrm{dx} + \int_3^6 (x-3)\ \mathrm{dx}$$

$$\int_0^3 -(x-3)\ \mathrm{dx} = \left(3x - \frac{x^2}{2}\right)\Bigg|_0^3 = \left(9 - \frac{3^2}{2}\right) - \left(0 - \frac{0^2}{2}\right) = \frac{9}{2}$$

$$\int_3^6 (x-3)\ \mathrm{dx} = \left(\frac{x^2}{2} - 3x\right)\Bigg|_3^6 = \left(\frac{6^2}{2} - 3\cdot 6\right) - \left(3 - \frac{3^2}{2}\right) = 0 - \left(\frac{-9}{2}\right) = \frac{9}{2}$$

$$\int_0^6 |x-3|\ \mathrm{dx} = \frac{9}{2} + \frac{9}{2} = 9$$

Problem 7. $\int_{-2}^{2} \frac{4+\sin(4x)}{4+x^2}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\pi}$

We begin by splitting the integral as follows.

$$\int_{-2}^{2} \frac{4+\sin(4x)}{4+x^2}\,\mathrm{dx} = \int_{-2}^{2} \frac{4}{4+x^2}\,\mathrm{dx} + \int_{-2}^{2} \frac{\sin(4x)}{4+x^2}\,\mathrm{dx}$$

The former integral looks similar to that of the derivative of $\arctan(x)$, but we modify as follows.

$$\int_{-2}^{2} \frac{4}{4+x^2} \cdot \frac{\frac{1}{4}}{\frac{1}{4}}\,\mathrm{dx} = \int_{-2}^{2} \frac{1}{1+\frac{x^2}{4}}\,\mathrm{dx} = \int_{-2}^{2} \frac{1}{1+\left(\frac{x}{2}\right)^2}\,\mathrm{dx}$$

This integral evaluates to the following.

$$\int_{-2}^{2} \frac{1}{1+\left(\frac{x}{2}\right)^2}\,\mathrm{dx} = 2\cdot\arctan\left(\frac{x}{2}\right)\Bigg|_{-2}^{2} = 2\arctan(1) - 2\arctan(-1) = 2\cdot\frac{\pi}{4} - 2\cdot\frac{-\pi}{4} = \pi$$

However, the latter integrand is odd (due to $\sin(4x)$), which means that the integral becomes 0 across symmetric bounds.

$$\int_{-2}^{2} \frac{\sin(4x)}{4+x^2}\,\mathrm{dx} = 0$$

Therefore, our initial integral becomes the following.

$$\int_{-2}^{2} \frac{4+\sin(4x)}{4+x^2}\,\mathrm{dx} = \pi + 0 = \boxed{\pi}$$

Problem 8. $\int_{0}^{1} \frac{1}{1+\frac{1}{1+\frac{1}{x}}}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{1}{2}\cdot\left(1+\frac{\ln(3)}{2}\right)}$

We first seek to simplify this nested fraction as much as possible, by making a common denominator in the most nested fraction as follows.

$$\int_{0}^{1} \frac{1}{1+\frac{1}{\frac{x}{x}+\frac{1}{x}}}\,\mathrm{dx} = \int_{0}^{1} \frac{1}{1+\frac{1}{\frac{x+1}{x}}}\,\mathrm{dx} = \int_{0}^{1} \frac{1}{1+\frac{x}{x+1}}\,\mathrm{dx}$$

We repeat the same procedure once again to simplify.

$$\int_{0}^{1} \frac{1}{\frac{x+1}{x+1}+\frac{x}{x+1}}\,\mathrm{dx} = \int_{0}^{1} \frac{1}{\frac{2x+1}{x+1}}\,\mathrm{dx} = \int_{0}^{1} \frac{x+1}{2x+1}\,\mathrm{dx}$$

We now formulate the numerator to take on the form of the denominator as follows.

$$x+1=\frac{2x+2}{2}=\frac{2x+1}{2}+\frac{1}{2}$$

Substituting this into the integrand and splitting, we have the following.

$$\int_0^1 \frac{\frac{2x+1}{2}+\frac{1}{2}}{2x+1}\ \mathrm{dx}=\int_0^1\left(\frac{2x+1}{2\cdot(2x+1)}+\frac{1}{2\cdot(2x+1)}\right)\ \mathrm{dx}=\int_0^1\left(\frac{1}{2}+\frac{1}{2\cdot(2x+1)}\right)\ \mathrm{dx}$$

Factoring out the common factor, we can now regularly integrate as follows.

$$\frac{1}{2}\cdot\left(\int_0^1\left(1+\frac{1}{2x+1}\right)\ \mathrm{dx}\right)=\frac{1}{2}\cdot\left(x+\frac{\ln|2x+1|}{2}\right)\Bigg|_0^1=\frac{1}{2}\cdot\left(1+\frac{\ln|2\cdot 1+1|}{2}\right)-\frac{1}{2}\cdot\left(0+\frac{\ln|2\cdot 0+1|}{2}\right)$$

Simplifying, we have the following.

$$\frac{1}{2}\cdot\left(1+\frac{\ln|2+1|}{2}\right)-\frac{1}{2}\cdot\left(\frac{\ln|0+1|}{2}\right)=\boxed{\frac{1}{2}\cdot\left(1+\frac{\ln(3)}{2}\right)}$$

Problem 9. $\displaystyle\int_e^{e^2}\left(\ln(\ln(x))+\frac{1}{\ln(x)}\right)\ \mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{e^2\ln(2)}$

To remove the nested natural logarithm, we consider the substitution $x=e^u$, which makes $\mathrm{dx}=e^u\ \mathrm{du}$. Thus, our integral becomes the following.

$$\int_e^{e^2}\left(\ln(\ln(x))+\frac{1}{\ln(x)}\right)\ \mathrm{dx}=\int_1^2\left(\ln(\ln(e^u))+\frac{1}{\ln(e^u)}\right)\cdot(e^u\ \mathrm{du})=\int_1^2\left(\ln(u)+\frac{1}{u}\right)\cdot(e^u\ \mathrm{du})$$

Note that our bounds were transformed as follows.

$$u=\ln(e)=1 \qquad\qquad u=\ln(e^2)=2$$

Splitting the integral into two, we have the following equivalent form.

$$\int_1^2\left(\ln(u)+\frac{1}{u}\right)\cdot(e^u\ \mathrm{du})=\int_1^2 e^u\cdot\ln(u)\ \mathrm{du}+\int_1^2\frac{e^u}{u}\ \mathrm{du}$$

Since we do not know how to evaluate the latter integral, we evaluate the former using integration by parts, in which we have the following.

$$a=\ln(u) \qquad\qquad \mathrm{da}=\frac{1}{u}\ \mathrm{du}$$

$$\mathrm{db}=e^u\ \mathrm{du} \qquad\qquad b=e^u$$

Therefore, our integral becomes the following.

$$\int_1^2 e^u\cdot\ln(u)\ \mathrm{du}=\int a\ \mathrm{db}=a\cdot b\Big|_1^2-\int_1^2 b\ \mathrm{da}=e^u\ln(u)\Big|_1^2-\int_1^2\frac{e^u}{u}\ \mathrm{du}$$

This exactly cancels out with the latter integral, which allows to evaluate to get our final answer as follows.

$$\int_1^2 \left(\ln(u)+\frac{1}{u}\right)\cdot(e^u\,\mathrm{du}) = \left(e^u\ln(u)\Big|_1^2 - \int_1^2 \frac{e^u}{u}\,\mathrm{du}\right) + \int_1^2 \frac{e^u}{u}\,\mathrm{du} = e^u\ln(u)\Big|_1^2 = e^2\ln(2)-e^1\ln(1) = \boxed{e^2\ln(2)}$$

Problem 10. $\int_1^2 \frac{x^3+x^{-3}}{x^1+x^{-1}}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{11}{6}}$

Let us recall the expansion of the sum of cubes as follows.

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

If we substitute $a = x$ and $b = x^{-1}$, we have the following.

$$\left(x+x^{-1}\right)^3 = x^3+3x^2\cdot x^{-1}+3x\cdot\left(x^{-1}\right)^2+\left(x^{-1}\right)^3$$

Simplifying, we have the following.

$$\left(x+x^{-1}\right)^3 = x^3+3x+3x^{-1}+x^{-3}$$

Solving for x^3+x^{-3}, we have the following.

$$\left(x+x^{-1}\right)^3 - 3\cdot\left(x+x^{-1}\right) = x^3+x^{-3}$$

Substituting this as the equivalent form of our integrand, we get the following.

$$\int_1^2 \frac{x^3+x^{-3}}{x^1+x^{-1}}\,\mathrm{dx} = \int_1^2 \frac{\left(x+x^{-1}\right)^3-3\cdot\left(x+x^{-1}\right)}{x^1+x^{-1}}\,\mathrm{dx} = \int_1^2 \left(\left(x+x^{-1}\right)^2-3\right)\,\mathrm{dx}$$

We can now freely expand as follows.

$$\int_1^2 \left(\left(x+x^{-1}\right)^2-3\right)\,\mathrm{dx} = \int_1^2 \left(x^2+2\cdot x\cdot x^{-1}+(x^{-1})^2-3\right)\,\mathrm{dx} = \int_1^2 \left(x^2+x^{-2}-1\right)\,\mathrm{dx}$$

Integrating regularly, we have the following.

$$\int_1^2 \left(x^2+x^{-2}-1\right)\,\mathrm{dx} = \left(\frac{x^3}{3}+\frac{x^{-1}}{-1}-x\right)\Big|_1^2 = \left(\frac{2^3}{3}-2^{-1}-2\right)-\left(\frac{1^3}{3}-1^{-1}-1\right)$$

Simplifying, we have the following.

$$\left(\frac{2^3}{3}-2^{-1}-2\right)-\left(\frac{1^3}{3}-1^{-1}-1\right) = \left(\frac{8}{3}-\frac{1}{2}-2\right)-\left(\frac{1}{3}-1-1\right) = \frac{8}{3}-\frac{1}{2}-\frac{1}{3} = \frac{7}{3}-\frac{1}{2} = \frac{14-3}{6} = \boxed{\frac{11}{6}}$$

Problem 11. $\int_0^3 \frac{3x+4}{x^2+4x+3}\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\frac{7\ln(2)}{2}}$

We first factor the denominator as follows.

$$\int_0^3 \frac{3x+4}{x^2+4x+3}\,\mathrm{dx} = \int_0^3 \frac{3x+4}{(x+1)\cdot(x+3)}\,\mathrm{dx}$$

We can then seek the partial fraction decomposition form of this fraction as follows.

$$\int_0^3 \frac{3x+4}{(x+1)\cdot(x+3)}\,\mathrm{dx} = \int_0^3 \left(\frac{A}{x+1}+\frac{B}{x+3}\right)\,\mathrm{dx}$$

To solve for A and B, we have the following system of equations in matching the numerator.

$$A\cdot(x+3)+B\cdot(x+1) = 3x+4$$

This gives us the following two equations.

$$A+B=3$$

$$3A+B=4$$

From these equations, we can deduce that $A=1/2$ and $B=5/2$. Therefore, our equivalent integrand becomes the following.

$$\int_0^3 \frac{3x+4}{(x+1)\cdot(x+3)}\,\mathrm{dx} = \int_0^3 \left(\frac{1}{2\cdot(x+1)}+\frac{5}{2\cdot(x+3)}\right)\,\mathrm{dx} = \frac{1}{2}\cdot\left(\int_0^3 \left(\frac{1}{x+1}+\frac{5}{x+3}\right)\,\mathrm{dx}\right)$$

We can evaluate this integral as follows.

$$\frac{1}{2}\cdot\left(\int_0^3 \left(\frac{1}{x+1}+\frac{5}{x+3}\right)\,\mathrm{dx}\right) = \frac{1}{2}\cdot(\ln|x+1|+5\cdot\ln|x+3|)\Big|_0^3 = \frac{1}{2}\cdot(\ln(4)+5\cdot\ln(6)) - \frac{1}{2}\cdot(\ln(1)+5\cdot\ln(3))$$

This simplifies as follows.

$$\frac{1}{2}\cdot(\ln(4)+5\cdot\ln(6)) - \frac{1}{2}\cdot(\ln(1)+5\cdot\ln(3)) = \frac{\ln(4)}{2}+\frac{5\cdot(\ln(2)+\ln(3))}{2}-\frac{5\ln(3)}{2} = \frac{2\ln(2)}{2}+\frac{5\ln(2)}{2} = \boxed{\frac{7\ln(2)}{2}}$$

Problem 12. $\int_0^1 e^x\cdot(\tan(x)+\tan^2(x)-x)\,\mathrm{dx}$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{e\cdot(\tan(1)-1)}$

We seek to manipulate the integrand in the form of a product rule, otherwise known as $(fg)' = f'g+g'f$.

Since we see that there is a e^x present in the integrand, we claim that $f = e^x$ since it will not disappear in the product rule. Substituting this into our product rule equation, we have the following.

$$(e^x \cdot g(x))' = e^x \cdot g(x) + e^x \cdot g'(x) = e^x \cdot (g(x) + g'(x))$$

Therefore, setting this equal to the integrand, we have the following.

$$e^x \cdot (g(x) + g'(x)) = e^x \cdot (\tan(x) + \tan^2(x) - x)$$
$$g(x) + g'(x) = \tan(x) + \tan^2(x) - x$$

We first recognize that we have the following relation.

$$(\tan(x))' = \sec^2(x)$$

Therefore, we use the Pythagorean identity to expand our functional equation as follows.

$$g(x) + g'(x) = \tan(x) + (\sec^2(x) - 1) - x$$

Regrouping, we have the following.

$$g(x) + g'(x) = (\tan(x) - x) + (\sec^2(x) - 1)$$

Therefore, we say that $g(x) = \tan(x) - x$. From there, our integrand becomes the following.

$$\int_0^1 e^x \cdot (\tan(x) + \tan^2(x) - x)\ \mathrm{dx} = \int_0^1 (e^x \cdot (\tan(x) - x))'\ \mathrm{dx} = (e^x \cdot (\tan(x) - x))\Big|_0^1$$

Evaluating, we have the final answer as follows.

$$(e^x \cdot (\tan(x) - x))\Big|_0^1 = (e^1 \cdot (\tan(1) - 1)) - (e^0 \cdot (\tan(0) - 0)) = e \cdot (\tan(1) - 1) - 1 \cdot (0 - 0) = \boxed{e \cdot (\tan(1) - 1)}$$

Problem 13. $\displaystyle\int_0^{\pi/4} \tan^2(2\theta - 1)\ \mathrm{d}\theta$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{\csc(2) - \frac{\pi}{4}}$

We immediately seek to eliminate the inner argument of the trigonometric function, so we consider the u-substitution of $u = 2\theta - 1$, which produces a differential of $\mathrm{du} = 2\ \mathrm{d}\theta$. Therefore, our integral becomes the following.

$$\int_0^{\pi/4} \tan^2(2\theta - 1)\ \mathrm{d}\theta \implies \int_{-1}^{(\pi-2)/2} \tan^2(u)\ \left(\frac{\mathrm{du}}{2}\right)$$

Note that our bounds were transformed as follows.

$$u = 2 \cdot 0 - 1 = 0 - 1 = -1$$

$$u = 2 \cdot \left(\frac{\pi}{4}\right) - 1 = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

From here, we can use the Pythagorean identity relation of trigonometric functions, which states that $\tan^2(u) + 1 = \sec^2(u)$. This also allows us to exploit that $\sec^2(u)$ has a nice anti-derivative, which we show as follows.

$$\int_{-1}^{(\pi-2)/2} \tan^2(u) \left(\frac{\mathrm{du}}{2}\right) = \frac{1}{2} \cdot \left(\int_{-1}^{(\pi-2)/2} (\sec^2(u) - 1)\, \mathrm{du}\right) = \frac{1}{2} \cdot \left[(\tan(u) - u) \Big|_{-1}^{(\pi-2)/2} \right]$$

Evaluating this, we have the following.

$$\frac{1}{2} \cdot \left[(\tan(u) - u) \Big|_{-1}^{(\pi-2)/2} \right] = \frac{1}{2} \cdot \left[\left(\tan\left(\frac{\pi - 2}{2}\right) - \left(\frac{\pi - 2}{2}\right) \right) - (\tan(-1) - (-1)) \right]$$

Simplifying, we have the following.

$$\frac{1}{2} \cdot \left[\left(\tan\left(\frac{\pi}{2} - 1\right) - \left(\frac{\pi}{2} - 1\right) \right) - (-\tan(1) + 1) \right] = \frac{1}{2} \cdot \left[\tan\left(\frac{\pi}{2} - 1\right) - \frac{\pi}{2} + \tan(1) \right]$$

We can simplify the first component using the definition of tangent and sum-angle identities as follows.

$$\tan\left(\frac{\pi}{2} - 1\right) = \frac{\sin\left(\frac{\pi}{2} - 1\right)}{\cos\left(\frac{\pi}{2} - 1\right)} = \frac{\sin\left(\frac{\pi}{2}\right)\cos(1) - \cos\left(\frac{\pi}{2}\right)\sin(1)}{\cos\left(\frac{\pi}{2}\right)\cos(1) + \sin\left(\frac{\pi}{2}\right)\sin(1)} = \frac{\sin\left(\frac{\pi}{2}\right)\cos(1)}{\sin\left(\frac{\pi}{2}\right)\sin(1)} = \frac{\cos(1)}{\sin(1)} = \cot(1)$$

Furthermore, we can simplify $\cot(1) + \tan(1)$ as follows using the Pythagorean identities and double-angle identities.

$$\cot(1) + \tan(1) = \frac{\cos(1)}{\sin(1)} + \frac{\sin(1)}{\cos(1)} = \frac{\cos^2(1) + \sin^2(1)}{\sin(1)\cos(1)} = \frac{1}{\frac{\sin(2)}{2}} = 2\csc(2)$$

Therefore, our final answer becomes the following.

$$\frac{1}{2} \cdot \left[\tan\left(\frac{\pi}{2} - 1\right) - \frac{\pi}{2} + \tan(1) \right] = \frac{1}{2} \cdot \left[2\csc(2) - \frac{\pi}{2} \right] = \boxed{\csc(2) - \frac{\pi}{4}}$$

Problem 14. $\displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{2} - \cos(\theta)}\, \mathrm{d}\theta$

Proposed by Ritvik Teegavarapu

Solution: $\boxed{2 \cdot \arctan(\sqrt{2} + 1)}$

We utilize the Weierstrass substitution to give us the following.

$$\int_0^{\pi/2} \frac{1}{\sqrt{2} - \cos(\theta)}\, \mathrm{d}\theta = \int_0^1 \frac{1}{\sqrt{2} - \frac{1-t^2}{1+t^2}} \cdot \left(\frac{2\, \mathrm{dt}}{1 + t^2}\right) = \int_0^1 \frac{2}{\sqrt{2}(1 + t^2) - (1 - t^2)}\, \mathrm{dt} = \int_0^1 \frac{2}{t^2 \cdot (\sqrt{2} + 1) + (\sqrt{2} - 1)}\, \mathrm{dt}$$

From here, we scale to then try and formulate an inverse trigonometric expression.

$$\int_0^1 \frac{2}{t^2 \cdot (\sqrt{2} + 1) + (\sqrt{2} - 1)} \cdot \frac{\sqrt{2} - 1}{\sqrt{2} - 1}\, \mathrm{dt} = \int_0^1 \frac{2(\sqrt{2} - 1)}{t^2 \cdot (\sqrt{2} + 1) \cdot (\sqrt{2} - 1) + (\sqrt{2} - 1)^2}\, \mathrm{dt} = \int_0^1 \frac{2(\sqrt{2} - 1)}{t^2 + (\sqrt{2} - 1)^2}\, \mathrm{dt}$$

Note that the inverse trigonometric function of interest here is as follows, which should be a fairly obvious result.

$$\int \frac{a}{b^2+x^2}\,\mathrm{dx} = \frac{a}{b}\cdot\arctan\left(\frac{x}{b}\right)+C$$

Using this, we have the following.

$$\int_0^1 \frac{2(\sqrt{2}-1)}{t^2+(\sqrt{2}-1)^2}\,\mathrm{dt} = \frac{2(\sqrt{2}-1)}{(\sqrt{2}-1)}\cdot\arctan\left(\frac{x}{\sqrt{2}-1}\right)\Bigg|_0^1 = 2\cdot\arctan\left(\frac{1}{\sqrt{2}-1}\right) = \boxed{2\cdot\arctan(\sqrt{2}+1)}$$